Week 5 Friday

February 7, 2020 10:18 AM

3.3 The Inverse of a Matrix Cont'd

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the <u>determinant</u> is

$\det A = ad - bc$

FACT

$$A^{-1} = \begin{cases} \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} & \text{if } \det A \neq 0 \\ \text{undefined} & \text{if } \det A = 0 \end{cases}$$

<u>Ex</u>: Find A^{-1}

a) $A = \begin{bmatrix} 1 & -4 \\ 7 & 2 \end{bmatrix}$

$\det A = 1(2) - (-4)(7) = 30$

$A^{-1} = \frac{1}{30}\begin{bmatrix} 2 & 4 \\ -7 & 1 \end{bmatrix}$

Check: $A^{-1}A = \frac{1}{30}\begin{bmatrix} 2 & 4 \\ -7 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 7 & 2 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 30 & 0 \\ 0 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

b) $A = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$

$\det A = 3(6) - (-2)(-9) = 0$

A^{-1} does not exist

(A is not invertible)

(A^{-1} is undefined)

System of equations: $A\vec{x} = \vec{b}$

If A^{-1} exists:

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

FACT

If A^{-1} exists then the system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$

Ex: Solve using A^{-1}

$$\begin{cases} 4x - 5y = -6 \\ -5x + 6y = 7 \end{cases}$$

$$A = \begin{bmatrix} 4 & -5 \\ -5 & 6 \end{bmatrix}$$

$\det A = -1$

$$A^{-1} = -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$= -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

$$= -\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \left(\begin{matrix} x = 1 \\ y = 2 \end{matrix} \right)$$

Finding A^{-1} for $n \times n$ matrices

$$[A \mid I] \xrightarrow{\text{row operations}} [I \mid A^{-1}]$$

Ex: Find A^{-1} for $A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 2R_1$, $R_3 - 2R_1$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & -2 & -2 & 0 & -2 & 1 \end{array}\right]$$

$R_1 - 2R_2$, $R_3 + 2R_2$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -8 & 2 & -6 & 1 \end{array}\right]$$

$\frac{R_3}{-8}$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{-2}{8} & \frac{6}{8} & \frac{-1}{8} \end{array}\right]$$

$-2 - 8\left(\frac{-2}{8}\right)$

$R_1 - 8R_3$, $R_2 + 3R_3$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & \frac{2}{8} & \frac{2}{8} & -\frac{3}{8} \\ 0 & 0 & 1 & \frac{-2}{8} & \frac{6}{8} & \frac{-1}{8} \end{array}\right]$$

$1 + 3\left(\frac{-2}{8}\right)$

$\underbrace{\hphantom{xxxxxxxx}}_{A^{-1}}$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 0 & -8 & 8 \\ 2 & 2 & -3 \\ -2 & 6 & -1 \end{bmatrix}$$

Ex: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 6 \\ 2 & 3 & 11 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 6 & 0 & 1 & 0 \\ 2 & 3 & 11 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$R_3 - R_2 \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 0 & & & \end{array}\right] \quad A^{-1} \text{ d.n.e.}$$

Fact

If a zero row appears on the left, A^{-1} does not exist

3 Properties of A^{-1}

① If A^{-1} exists, $\quad (A^{-1})^{-1} = A$

② $(A^{-1})^T = (A^T)^{-1}$

<u>Ex</u>: check for $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

$$(A^{-1})^T = \left(\frac{1}{1}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} \checkmark$$

③ $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$ ★

In particular $(AB)^{-1} = B^{-1}A^{-1}$

Ex: A: putting on socks
B: " shoes

$$(AB)^{-1} = B^{-1}A^{-1}$$

Undo operations, in reverse order

Comment: $(A^n)^{-1} = (A^{-1})^n$

We can write A^{-n} without confusion