3.1 Matrix Operations Cont'd

Ex: Write as a matrix equation

$$\begin{cases} 3x - 7y = 4 \\ -2x + y = -11 \end{cases}$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 3 & -7 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \end{bmatrix}$$

Ex: A : test marks

$$\begin{array}{c} \\ T1 \\ T2 \\ \text{Exam} \end{array} \begin{array}{c} \begin{array}{cc} \text{Al} & \text{Bob} \end{array} \\ \begin{bmatrix} 50 & 60 \\ 90 & 80 \\ 75 & 70 \end{bmatrix} \end{array}$$

B : weightings

$$\begin{array}{ccc} T1 & T2 & \text{Exam} \\ \end{array}$$
$$\begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Find Al and Bob's final grades

→ Need compatible sizes, categories

$$BA = \begin{array}{ccc} T1 & T2 & E \end{array} \quad \begin{array}{c} T1 \\ T2 \\ E \end{array}$$
$$\begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.2 & 0.6 \end{bmatrix} \begin{array}{c} \begin{array}{cc} \text{Al} & \text{Bob} \end{array} \\ \begin{bmatrix} 50 & 60 \\ 90 & 80 \\ 75 & 70 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} \text{Al} & \text{Bob} \end{array} \\ \begin{bmatrix} 73 & 70 \end{bmatrix} \end{array}$$

Ex: Find 2nd column of

$$\begin{bmatrix} 1 & 2 \\ 2 & 6 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 4 & 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} & 16 & \\ & 46 & \\ & 65 & \end{bmatrix}$$

Notice $\begin{bmatrix} 16 \\ 46 \\ 65 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$

Fact: Columns of AB are linear combinations of the columns of A

Useful in Section 7.3

Outer Product Expansion of AB

$$\left[A_1 \,\middle|\, A_2 \,\middle|\, A_3 \,\middle|\, \cdots \right] \begin{bmatrix} B_1 \\ \hline B_2 \\ \hline B_3 \\ \hline \vdots \end{bmatrix}$$

$$AB = A_1B_1 + A_2B_2 + A_3B_3 + \cdots$$

Ex: Find the outer product expansion of AB

$$A = \left[\begin{array}{c|c} 1 & 3 \\ 0 & -2 \end{array}\right] \qquad B = \left[\begin{array}{cc} 1 & 7 \\ \hline 4 & 2 \end{array}\right]$$

$$AB = A_1B_1 + A_2B_2$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 7 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} \begin{bmatrix} 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 12 & 6 \\ -8 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 13 \\ -8 & -4 \end{bmatrix}$$

Check: $AB = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 13 \\ -8 & -4 \end{bmatrix}$ ✓

Useful in Ch. 5

Powers of a Matrix

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$A^2 = AA$$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$$

Usual exponent rules apply

$$A^3 = A^2 A \quad \text{or} \quad A^3 = A A^2$$

$$A^5 A^7 = A^{12}$$

$$(A^3)^4 = A^{12}$$

$$\boxed{\textbf{Fact:} \quad AI = A \qquad IA = A}$$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ etc.

Parallel to $7(1) = 7$

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ ✓

Ex: Simplify B^{2018} given that $B^3 = I$

B^{2018}

$$\boxed{\begin{aligned} 2018 &= ?(3) + ? \\ \frac{2018}{3} &\approx 672.7 \\ 2018 &= 672(3) + ? \\ 2018 &= 672(3) + 2 \end{aligned}}$$

$$\begin{aligned} &= B^{672(3)+2} \\ &= B^{672(3)} B^2 \\ &= (B^3)^{672} B^2 \\ &= I^{672} B^2 \\ &= I B^2 \\ &= B^2 \end{aligned}$$

Alternatively: $B^{2018 \bmod 3}$

Def

Let $\mathbf{0}$ be the zero matrix

$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ etc.

Def

Let $\mathbf{0}$ be the zero matrix

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ etc.}$$

<u>Ex</u>: Find a 2x2 matrix A so that $A^2 = \mathbf{0}$ but $A \neq \mathbf{0}$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$