2.3 Span and Linear Independence Cont'd

<u>Ex</u>: Are $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\2\end{bmatrix}$ linearly independent?

Let $c_1\begin{bmatrix}1\\1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\0\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \end{array}$$

$$R_2 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\frac{R_2}{-1} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1 - R_2 \\ \\ R_3 + R_2 \end{array} \quad \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \end{array} \quad \text{REF}$$

$c_1 = 0$

$c_2 = 0$

$2c_3 = 0 \rightarrow c_3 = 0$

$\boxed{\text{Yes}}$

If $c_1 = c_2 = c_3 = 0$ is the <u>only</u> solution, then vectors are lin. ind.

FACT

A set of $>n$ vectors in $\mathbb{R}^n$ is linearly dependent

Ex: $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}2\\7\end{bmatrix}$ are linearly dependent.

Why? Let $c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = \vec{0}$

$$n\left\{ \begin{array}{c} c_1 \; c_2 \; \cdots \; c_m \\ \underbrace{\left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{array}\right]}_{>n} \end{array} \right.$$

System has ∞-many solutions (Section 2.2)

Ex: Find a linear dependence relationship (linear dependency) among

$\begin{bmatrix}1\\6\end{bmatrix}, \begin{bmatrix}2\\6\end{bmatrix}, \begin{bmatrix}4\\30\end{bmatrix}$

Method I Let $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 6 & 6 & 30 & 0 \end{array}\right]$$

$$R_2 - 6R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right]$$

$$R_2 - 6R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right]$$

$$\frac{R_2}{-6} \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_1 - 2R_2 \quad \begin{array}{cccc} c_1 & c_2 & c_3 & \\ \hline \end{array}$$

$$\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad \text{RREF}$$

$c_3 = t$

$$c_1 + 6c_3 = 0 \rightarrow \quad c_1 = -6t$$

$$c_2 = t$$

Use any nonzero t-value:
$(t=1)$

$$c_1 = -6 \qquad c_2 = 1 = c_3$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$-6\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$$

Method II

Vectors $\rightarrow$ rows

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{array} \begin{bmatrix} 1 & 6 \\ 2 & 6 \\ 4 & 30 \end{bmatrix}$$

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 - 2\vec{v}_1 \\ \vec{v}_3 - 4\vec{v}_1 \end{array} \begin{bmatrix} 1 & 6 \\ 0 & -6 \\ 0 & 6 \end{bmatrix}$$

$$\underbrace{(\vec{v}_3 - 4\vec{v}_1) + (\vec{v}_2 - 2\vec{v}_1)}_{-6\vec{v}_1 + \vec{v}_2 + \vec{v}_3} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\boxed{-6\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}}$$

Any zero row gives a linear dependency.

Method I describes the <u>general</u> dependency.
Method II gives 1 <u>specific</u> dependency

<u>Preview of Section 3.5</u>

The smallest set of vectors that describe a geometric object?

Need 1) span(vectors) = object
2) vectors to be linearly independent (no redundancy)

<u>2.4 Applications</u>

① Find the parabola through $(1,-2)$, $(-1,8)$ and $(2,-1)$.

Parabola $\quad y = ax^2 + bx + c$

Variables: a, b, c

$x = 1$, $y = -2$: $\quad -2 = a + b + c \quad (1)$

$x = -1$, $y = 8$: $\quad 8 = a - b + c \quad (2)$

$x = 2$, $y = -1$: $\quad -1 = 4a + 2b + c \quad (3)$

$$\begin{array}{c} \begin{array}{cccc} a & b & c & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 1 & -1 & 1 & 8 \\ 4 & 2 & 1 & -1 \end{array}\right] \end{array}$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cccc} a & b & c & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array} \quad \text{RREF}$$

$a = 2$
$b = -5$
$c = 1$

$y = ax^2 + bx + c$

$$\boxed{y = 2x^2 - 5x + 1}$$

$$\boxed{y = 2x^2 - 5x + 1}$$