Week 3 Friday

January 24, 2020 9:38 AM

2.2 Solving Systems Cont'd

Ex: Consider $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right]$ or $\begin{cases} 1x + 2y = 0 \\ 3x + 4y = 0 \end{cases}$

It has solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Terminology

$\vec{x} = \vec{0}$ is called the trivial solution

$\left[\begin{array}{c|c} \text{anything} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]$ is called a homogeneous system

Fact

A homogeneous system always has at least 1 solution : the trivial solution

Ex: How many solutions to the system?

$$m\left\{\left[\begin{array}{c|c} \underbrace{\qquad\qquad\qquad\qquad}_{n} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right]\right.$$

Given $n > m$.

At least 1 solution (because it's homogeneous)

$$\left[\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{array}\right] \quad \text{REF}$$

$\uparrow$

. .

At least 1 parameter $\Rightarrow$ ∞-many solutions

2.3 Span and Linear Independence

Ex: Is $\begin{bmatrix} 8 \\ -10 \end{bmatrix}$ a linear combination of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

Let $\underbrace{c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix}} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$

$$\begin{bmatrix} -c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ -3c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} -c_1 + 2c_2 \\ 2c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$$

$$\begin{cases} -c_1 + 2c_2 = 8 \\ 2c_1 - 3c_2 = -10 \end{cases}$$

$$\begin{array}{ccc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 2 & 8 \\ 2 & -3 & -10 \end{array}\right]$$

$\frac{R_1}{-1}$ $\left[\begin{array}{cc|c} 1 & -2 & -8 \\ 2 & -3 & -10 \end{array}\right]$

$R_2 - 2R_1$ $\left[\begin{array}{cc|c} 1 & -2 & -8 \\ 0 & 1 & 6 \end{array}\right]$ REF

$$R_1 + 2R_2 \quad \begin{array}{cc|c} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right] \quad \text{RREF}$$

$c_1 = 4$
$c_2 = 6$

$\boxed{\text{Yes}}$

To check: $4\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 6\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$ ✓

Ex: Is $\bar{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ a linear combination of $\bar{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$?

Let $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{array}\right]$$

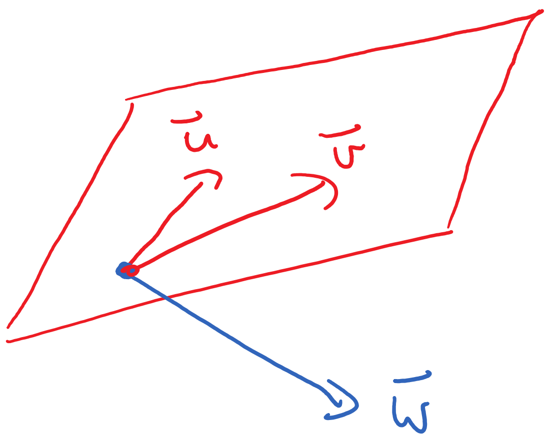
$$R_3 - R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{array}\right]$$

$$\frac{R_2}{3} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & -1 & 1 \end{array}\right]$$

$R_3 + R_2$ $\left[\begin{array}{cc|c} 0 & 0 & \frac{4}{3} \end{array}\right]$

No solution

$\boxed{\text{No}}$



Recall: A consistent system has at least 1 solution (is solvable)

FACT

$\vec{b}$ is a linear combination of the columns of A if and only if $[A \mid \vec{b}]$ is consistent

Def

The span of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is the set of all linear combinations of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

Ex: a) $\text{span}(\vec{a}, \vec{b}) = \{\vec{0}, \vec{a}, 3\vec{a}, -4\vec{a}, 2\vec{b}, -7\vec{b}, \vec{a}+\vec{b}, -11\vec{a}+8\vec{b}, \dots\}$

b) $\text{span}(\vec{u}_1, \dots, \vec{u}_n) = \{c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_n\vec{u}_n\}$

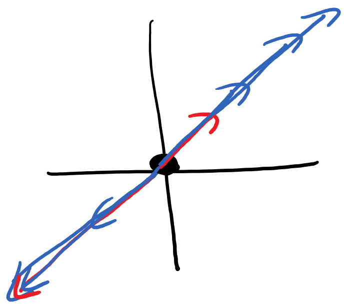
$c_1, \dots, c_n$: any real #

Fact

$\vec{0}$ is in every span

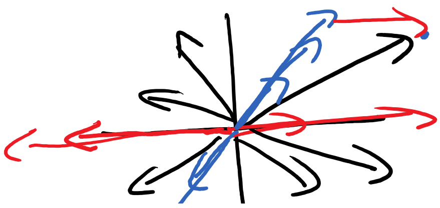
Ex: Describe the span geometrically

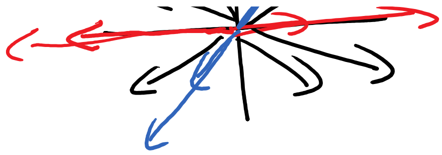
a) $\text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-3\\-3\end{bmatrix}\right)$



Line through origin in $\mathbb{R}^2$ with $\vec{d} = \begin{bmatrix}1\\1\end{bmatrix}$

b) $\text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-3\\0\end{bmatrix}\right)$



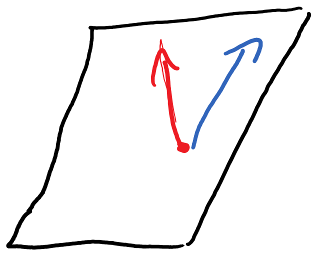


All of $\mathbb{R}^2$
(whole xy-plane)

c) $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 8 \end{bmatrix} \right)$

Line through origin in $\mathbb{R}^3$
with $\vec{d} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

d) $\text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$



Plane in $\mathbb{R}^3$ through origin
with direction vectors
$\vec{d}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

e) $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \mathbb{R}^3$