2.2 Solving Systems Cont'd

Ex: Solve by Gauss-Jordan Elimination

$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 10 & 5 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 3 & 4 & 12 & 9 \end{array}\right]$$

$R_3 - R_1$
$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 10 & 5 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 4 \end{array}\right]$$

$R_1 - R_2$, $R_3 - 2R_2$
$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 9 & 3 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF ✓

$y = s$, $z = t$ "parameters"

$w + y + 9z = 3 \rightarrow w = -y - 9z + 3 \rightarrow \boxed{w = -s - 9t + 3}$

$x + y + z = 2 \rightarrow x = 2 - y - z \rightarrow \boxed{x = 2 - s - t}$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -9 \\ -1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Ex: Find the intersection of the 2 lines:

$$\vec{x} = \begin{bmatrix} -5 \\ 6 \\ 5 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Not parallel

May or may not intersect in 3D

$$\vec{x} = \vec{x}$$

$$\begin{bmatrix}-5\\6\\5\end{bmatrix} + s\begin{bmatrix}2\\1\\-1\end{bmatrix} = \begin{bmatrix}-5\\4\\-1\end{bmatrix} + t\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$s\begin{bmatrix}2\\1\\-1\end{bmatrix} - t\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-5\\4\\-1\end{bmatrix} - \begin{bmatrix}-5\\6\\5\end{bmatrix}$$

$$s\begin{bmatrix}2\\1\\-1\end{bmatrix} + t\begin{bmatrix}-1\\-1\\-1\end{bmatrix} = \begin{bmatrix}0\\-2\\-6\end{bmatrix}$$

$$\begin{cases} 2s - t = 0 \\ \end{cases}$$

$$\begin{array}{cc} s & t \end{array}$$
$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 1 & -1 & -2 \\ -1 & -1 & -6 \end{array}\right]$$

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 2 & -1 & 0 \\ -1 & -1 & -6 \end{array}\right]$$

$R_2 - 2R_1$, $R_3 + R_1$
$$\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 0 & 1 & 4 \\ 0 & -2 & -8 \end{array}\right]$$

$R_1 + R_2$, $R_3 + 2R_2$
$$\begin{array}{cc} s & t \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \text{no info}$$

1 solution

$s = 2$
$t = 4$

Sub $s = 2 \to 1^{st}$ line

or Sub $t = 4 \to 2^{nd}$ line

$$\bar{x} = \begin{bmatrix} -5 \\ 6 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 3 \end{bmatrix}$$

Point $= (-1, 8, 3)$

Ex: How many solutions does the system have?

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right] \quad (x, y)$$

$$R_2 - kR_1 \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right]$$

$1-k^2 \neq 0$

$$\frac{R_2}{1-k^2} \quad \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{1-k}{1-k^2} \end{array}\right]$$

1 SOLUTION

$1-k^2 = 0$

$(1-k)(1+k) = 0$

$k = 1$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

∞-MANY SOLUTIONS

$k = -1$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

NO SOLUTION

$$\begin{cases} 1 \text{ solution} & \text{if } 1-k^2 \neq 0 \\ \infty\text{-many} & \text{if } k = 1 \\ 0 & \text{if } k = -1 \end{cases}$$

Def

Rank of a matrix: # of nonzero rows in REF or RREF

Fact

rank + (# parameters in solution) = # variables

True for all consistent systems (solvable)

Ex:

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 3 & 6 \\ 0 & \textcircled{4} & 5 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

$z = t$

rank = 2

parameters = 1

Fact, rephrased:

(# columns with circles) + (# columns without circles) = # columns