Week 2 Friday

January 17, 2020 9:35 AM

(1.4) Cross Product Cont'd

Ex: $\vec{u} = [1,2,3]$ $\vec{v} = [4,5,6]$

Find $\vec{u} \times \vec{v}$

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \end{matrix}$$

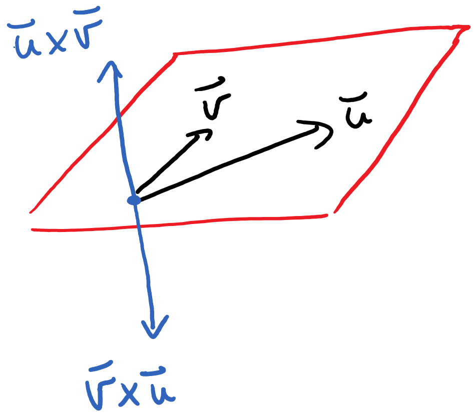
$\vec{u} \times \vec{v} = [2(6)-3(5), \ \ldots \ ]$

$= [-3, 6, -3]$

Facts

1) $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$

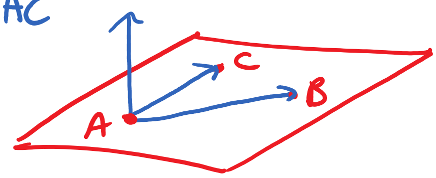
2) $\vec{u} \times \vec{v}$ is $\perp$ to $\vec{u}$ and $\vec{v}$



Right Hand Rule

Ex: Find the general form of the plane through $A = (1,3,6)$, $B = (2,1,4)$ and $C = (1,-1,5)$

$\vec{n} = \vec{AB} \times \vec{AC}$



$$\vec{AB} = [1, -2, -2]$$

$$\vec{AC} = [0, -4, -1]$$

$$\vec{n} = \vec{AB} \times \vec{AC} = [-6, 1, -4]$$

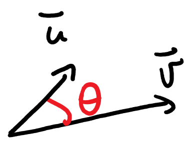
$$\boxed{\begin{array}{ccccc} 1 & -2 & -2 & 1 & -2 \\ 0 & -4 & -1 & 0 & -4 \end{array}}$$

Normal form $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

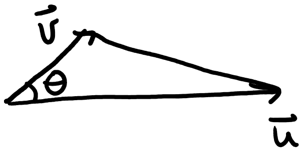
$$\begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

General form $\quad -6x + y - 4z = -27$

Recall $\quad \vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

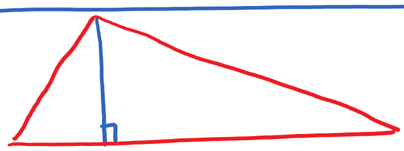


$$\boxed{\text{FACT} \quad \|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta}$$

Ex: Show that the area of the triangle  is $\frac{1}{2}\|\vec{u}\times\vec{v}\|$

$A = \frac{1}{2} b h$

$= \frac{1}{2}\|\vec{u}\|\, h$



$\sin\theta = \dfrac{h}{\|\vec{v}\|}$

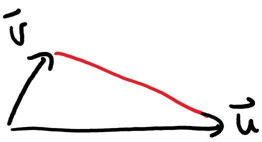
$h = \|\vec{v}\|\sin\theta$

$A = \frac{1}{2}\|\vec{u}\|\,\|\vec{v}\|\sin\theta$

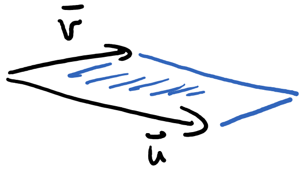
$= \frac{1}{2}\|\vec{u}\times\vec{v}\|$ ✓

FACT

$A(\text{triangle}) = \frac{1}{2}\|\vec{u}\times\vec{v}\|$

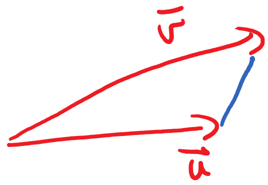


$A(\text{parallelogram}) = \|\vec{u}\times\vec{v}\|$



for $\vec{u}, \vec{v}$ in $\mathbb{R}^3$

Ex: Find the area of the triangle determined by $\vec{u} = [1,4,5]$ and $\vec{v} = [2,3,6]$



$$\begin{array}{ccccc} 1 & 4 & 5 & 1 & 4 \\ 2 & 3 & 6 & 2 & 3 \end{array}$$

$\vec{u} \times \vec{v} = [9, 4, -5]$

$\| \vec{u} \times \vec{v} \| = \sqrt{81 + 16 + 25} = \sqrt{122}$

Area $= \dfrac{\sqrt{122}}{2}$

$\boxed{\text{\underline{Matrix}: rectangular array}}$

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

<u>Size</u> of a matrix: (# rows) x (# columns)

e.g. A is 2×3

NOTATION

The <u>determinant</u> of A is written $\det A$ or $|A|$.

Only defined for square matrices.

FORMULAS

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"Cofactor Expansion"

Signs alternate $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Ex: Compute $\det \begin{bmatrix} -1 & -4 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 8 \end{bmatrix}$

$$\begin{vmatrix} -1 & -4 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 8 \end{vmatrix} = -1\begin{vmatrix} & \\ & \end{vmatrix} + 4\begin{vmatrix} & \\ & \end{vmatrix} + 6\begin{vmatrix} & \\ & \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ & & \end{bmatrix}$$

$$= -1\begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} + 6\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -1(6) + 4(6) + 6(0)$$

$$= 18$$

Notation

$$[1,0,0] = \vec{i}$$
$$[0,1,0] = \vec{j}$$
$$[0,0,1] = \vec{k}$$

2nd Method for Cross Product :

$$[2,1,3] \times [-6,4,2] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ -6 & 4 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 3 \\ -6 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix}$$

$$= \vec{i}(-10) - \vec{j}(22) + \vec{k}(14)$$

$$= [-10,0,0] + [0,-22,0] + [0,0,14]$$

$$= [-10,-22,14]$$

Gives same result as :

$$\begin{matrix} 2 & 1 & 3 & 2 & 1 \\ -6 & 4 & 2 & -6 & 4 \end{matrix}$$

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