Week 10 Friday

March 13, 2020 10:19 AM

4.4 Diagonalization Cont'd

Ex: Diagonalize $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$ (if possible)

$\lambda = 4, 4$ (A is lower triangular)

Find a basis for each eigenspace

E_4 : $[A - 4I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_2 = t$

$x_1 = 0$

eigenvectors $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$

basis for $E_4 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Not enough basis vectors to build P.

A cannot be diagonalized.

More detail: characteristic polynomial of $A = |A - \lambda I| = (4-\lambda)^2$

algebraic multiplicity of $\lambda = 4$ is 2

geometric " " is 1

geo. mult. < alg. mult. $\Rightarrow$ can't diagonalize A

Ex:

$$\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (-4)^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

FACT

If D is diagonal then D^n is diagonal, with n^{th} powers on the diagonal.

FACT

If $P^{-1}AP = D$ then $A^n = PD^nP^{-1}$

Why?

$$P^{-1}AP = D$$

$$P\cancel{P^{-1}}AP = PD$$

$$A\cancel{PP^{-1}} = PDP^{-1}$$

$$A^n = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$$

$$A^n = PD^nP^{-1}$$

Ex: $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizes A

to produce $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Find A^k

$$P^{-1}AP = D$$

$$\cancel{P P^{-1}} A \cancel{P P^{-1}} = PDP^{-1}$$

$$A^k = PD\cancel{P^{-1}} \cdots \cancel{P}DP^{-1}$$

$$A^k = PD^kP^{-1}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{[P|I] \rightsquigarrow [I|P^{-1}]}$$

$$= \begin{bmatrix} 3^k & 0 & 4^k \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^k & 0 & 4^k - 3^k \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \leftarrow -3^k + 4^k$$

__Ex:__ Application of A^n

Consider a company with 1,000 machines

W = a machine is working

B = " broken

Probability Matrix

$$\begin{array}{c|cc} & W & B \\ \hline W & 0.99 & 0.5 \\ B & 0.01 & 0.5 \end{array} = A$$

← today

↑ tomorrow

State Vector

$$\vec{x} = \begin{array}{c} W \\ B \end{array} \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

All machines are working now.

Tomorrow $\quad A\vec{x} = \begin{bmatrix} 0.99 & 0.5 \\ 0.01 & 0.5 \end{bmatrix} \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = \begin{bmatrix} 990 \\ 10 \end{bmatrix} \begin{array}{c} W \\ B \end{array}$

2 days from now $\quad A^2\vec{x} = \begin{bmatrix} 0.99 & 0.5 \\ 0.01 & 0.5 \end{bmatrix} \begin{bmatrix} 990 \\ 10 \end{bmatrix} \approx \begin{bmatrix} 985 \\ 15 \end{bmatrix} \begin{array}{c} W \\ B \end{array}$

As $n \to \infty$ $\quad A^n\vec{x} \to \begin{bmatrix} 980 \\ 20 \end{bmatrix}$

In the long run, 20 machines will be broken each day.