Week 10 Thursday

March 12, 2020 9:39 AM

4.3 Finding Eigenvalues Cont'd

Warm-Up: $A\begin{bmatrix}-1\\5\end{bmatrix} = 2\begin{bmatrix}-1\\5\end{bmatrix}$

Find $A^4\begin{bmatrix}-1\\5\end{bmatrix}$

$$= \lambda^4\begin{bmatrix}-1\\5\end{bmatrix}$$

$$= 16\begin{bmatrix}-1\\5\end{bmatrix}$$

$$= \begin{bmatrix}-16\\80\end{bmatrix}$$

Recalled Fact 5

Ex: A has eigenvector $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ corresponding to $\lambda_1 = -2$

and " $\vec{v}_2 = \begin{bmatrix}2\\-1\end{bmatrix}$ " $\lambda_2 = 3$

Find $A^3\begin{bmatrix}11\\2\end{bmatrix}$

① Let $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}11\\2\end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & -1 & 2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

$c_1 = 3 \quad c_2 = 4$

② $A^3\begin{bmatrix}11\\2\end{bmatrix} = A^3\left(3\begin{bmatrix}1\\2\end{bmatrix} + 4\begin{bmatrix}2\\-1\end{bmatrix}\right)$

$$= c_1\lambda_1^3\vec{v}_1 + c_2\lambda_2^3\vec{v}_2$$

$$= 3(-2)^3\begin{bmatrix}1\\2\end{bmatrix} + 4(3)^3\begin{bmatrix}2\\-1\end{bmatrix}$$

$$= -24 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 108 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 192 \\ -156 \end{bmatrix}$$

4.4 Diagonalization

Def

An $n \times n$ matrix A is diagonalizable if there exist an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$

Ex: Find P that diagonalizes $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\rightarrow$ Find a basis for each eigenspace

$\lambda = 2, 3, 3$ (A is upper triangular)

$\lambda = 2$: Solve $[A - 2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \rightsquigarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$\boxed{x_1 = t}$

$x_2 = 0$
$x_3 = 0$

Eigenvectors $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$

Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\lambda = 3$: Solve $[A - 3I \mid \vec{0}]$

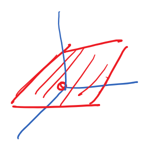
$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$\boxed{x_2 = s}$ $\boxed{x_3 = t}$

$x_1 + 2x_3 = 0 \rightarrow \boxed{x_1 = -2t}$

Eigenvectors $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$



Basis for $E_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

To build P: Put basis vectors into columns of P $\qquad P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

To build D: Put eigenvalues on diagonal, in same order as P.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To check: $P^{-1} A P = D$ ✓

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$[P \mid I] \rightsquigarrow [I \mid P^{-1}]$

FACT

A is diagonalizable if and only if geometric multiplicity = algebraic multiplicity for all eigenvalues.

$\dim(E_\lambda)$

Ex: Basis for $E_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Geo. Mult. of $\lambda = 3$ is 2

Ex: A had $\lambda = 2, 3, 3$

λ	Alg. Mult.
2	1
3	2

Rephrased: A is diagonalizable exactly when you have enough columns to make P square.

Quick Ex: $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$ $\lambda = 4, 4$ (A is lower triangular)

alg. mult. of $\lambda = 4$ is 2

Possible geo. mult. of $\lambda = 4$: 1 or 2

Can diagonalize A if geo. mult. = 2