Week 10 Tuesday

March 10, 2020 8:32 AM

Test 3
Thurs March 26^{th}

4.3 Finding Eigenvalues

Ex: Find all eigenvalues of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$

Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -4-\lambda & 3 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-4-\lambda)(7-\lambda) = 0$$

$$\lambda = 1, -4, 7$$

FACT

If A is diagonal or upper/lower triangular, the eigenvalues of A are the diagonal entries.

Ex: Find all eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 0 \qquad \begin{bmatrix} + & - & + \\ & & \\ & & \end{bmatrix}$$

$$(3-\lambda)\begin{vmatrix} -5-\lambda & 1 \\ -6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & 1 \\ 6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & -5-\lambda \\ 6 & -6 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-5-\lambda)(2-\lambda) + 6\right] + \left[7(2-\lambda) - 6\right] + \left[-42 - 6(-5-\lambda)\right] = 0$$

$$(3-\lambda)(\lambda^2 + 3\lambda - 4) \quad -7\lambda + 8 \quad + 6\lambda - 12 = 0$$

$$\boxed{\begin{array}{rrrr} & 3\lambda^2 & +9\lambda & -12 \\ -\lambda^3 & -3\lambda^2 & +4\lambda & \\ & & -7\lambda & +8 \\ + & & 6\lambda & -12 \\ \hline -\lambda^3 & & +12\lambda & -16 \end{array}}$$

$$-\lambda^3 + 12\lambda - 16 = 0$$

$$\boxed{\lambda^3 - 12\lambda + 16 = 0}$$

Integer Roots Theorem

If a polynomial has integer coefficients and the leading coefficient is 1, then any integer roots divide the constant.

Possible roots : $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

Check $\lambda = 1$: $1^3 - 12(1) + 16 = 0$? No

$\lambda = -1$: No

$\lambda = 2$: $2^3 - 12(2) + 16 = 0$ YES

$\lambda = 2$ is a solution

$\Rightarrow$ $(\lambda - 2)$ is a factor of polynomial

Long Division

$$\begin{array}{r|l} & \lambda^2 + 2\lambda - 8 \\ \hline \boxed{\lambda} - 2 & \boxed{\lambda^3} + 0\lambda^2 - 12\lambda + 16 \\ & -(\lambda^3 - 2\lambda^2) \\ \hline & 2\lambda^2 - 12\lambda + 16 \\ & -(2\lambda^2 - 4\lambda) \\ \hline & -8\lambda + 16 \\ & -(-8\lambda + 16) \\ \hline & 0 \end{array}$$

$$\lambda^3 - 12\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda^2 + 2\lambda - 8) = 0$$

$$(\lambda - 2)(\lambda + 4)(\lambda - 2) = 0$$

$$(\lambda - 2)^2(\lambda + 4) = 0$$

$$\lambda = 2, -4$$

Given bases for eigenspaces

$$E_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$E_{-4} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

To check:
Solve $[A - \lambda I \mid \vec{0}]$

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

FACT

When bases for different eigenspaces are combined, the new set is linearly independent.

e.g. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent

Useful in Section 4.4

Def

- Characteristic equation of A : $(\lambda-2)^2(\lambda+4)=0$

  (in general $|A-\lambda I|=0$)

- Algebraic multiplicity of an eigenvalue λ_i :

  exponent on $(\lambda-\lambda_i)$ in characteristic equation

- Geometric multiplicity of an eigenvalue :

  # of basis vectors in the eigenspace

Ex: $A=\begin{bmatrix}3 & -1 & 1\\ 7 & -5 & 1\\ 6 & -6 & 2\end{bmatrix}$

Given $(\lambda-2)^2(\lambda+4)=0$

$E_2 = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$

$E_{-4} = \text{span}\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)$

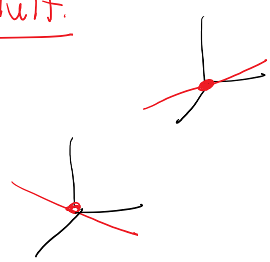
Find :

a) characteristic equation

$(\lambda-2)^2(\lambda+4)=0$

b) algebraic and geometric multiplicities of the eigenvalues

	Alg. Mult.	Geo. Mult.
$\lambda=2$	2	1
$\lambda=-4$	1	1



$\boxed{\text{geo. mult.} \le \text{alg. mult.}}$