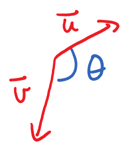
1.2 Length and Angle Cont'd

Fact

The angle θ between $\vec{u}$ and $\vec{v}$ is defined to be $0^\circ \leq \theta \leq 180^\circ$



Formula

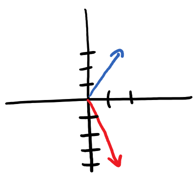
$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

Comments 1) In $\mathbb{R}^4$ and higher dimensions, this is a definition of θ

2) θ is always defined

<u>Ex</u>: Find the angle between $\vec{u} = [1,-4]$ and $\vec{v} = [2,3]$



Guess $\theta \simeq 135^\circ$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$-10 = \sqrt{17}\,\sqrt{13}\,\cos\theta$$

$\boxed{1(2) + -4(3)}$ $\qquad$ $\boxed{\sqrt{2^2+3^2}}$

$$\frac{-10}{\sqrt{17}\sqrt{13}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-10}{\sqrt{17}\sqrt{13}}\right)$$

$$\approx 132^\circ$$

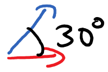
The sign of $\vec{u}\cdot\vec{v}$ gives some info about θ

$\vec{u}\cdot\vec{v} > 0$	$\vec{u}\cdot\vec{v} = 0$	$\vec{u}\cdot\vec{v} < 0$
$\cos\theta > 0$	$\cos\theta = 0$	$\cos\theta < 0$
$0^\circ \leq \theta < 90^\circ$	$\theta = 90^\circ$	$90^\circ < \theta \leq 180^\circ$

Suppose $\bar{u}$ and $\bar{v}$ are unit vectors (length 1)

$\bar{u}\cdot\bar{v} = \cos\theta$

$\bar{u}\cdot\bar{v} = 1$

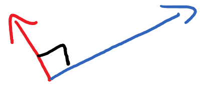


$\bar{u}\cdot\bar{v} = \frac{\sqrt{3}}{2}$

$\bar{u}\cdot\bar{v} = 0$

$\bar{u}\cdot\bar{v} = -1$

Def
$\bar{u}$ and $\bar{v}$ are <u>orthogonal</u> if $\bar{u}\cdot\bar{v} = 0$



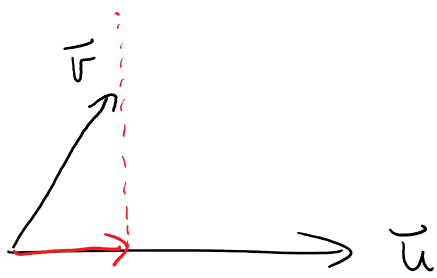
$\bar{u}$ and $\bar{v}$ are orthogonal (algebra language)
" perpendicular (geometry language)

The projection of $\bar{v}$ onto $\bar{u}$ is written $\text{proj}_{\bar{u}}\,\bar{v}$

Could be read as "projection onto $\bar{u}$ of $\bar{v}$"

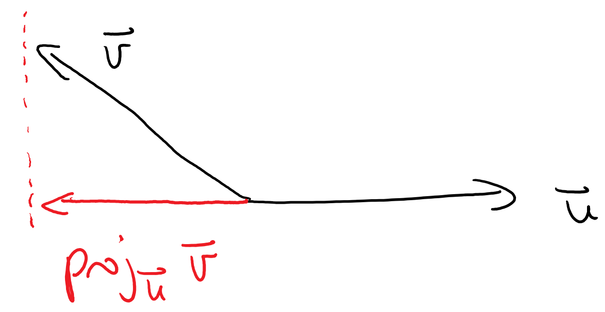
<u>Quick Ex</u>:

a)

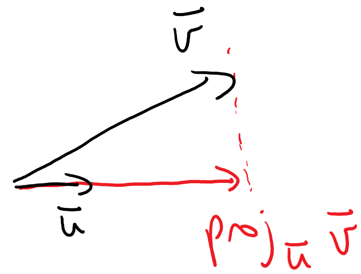


$\text{proj}_{\bar{u}}\,\bar{v}$

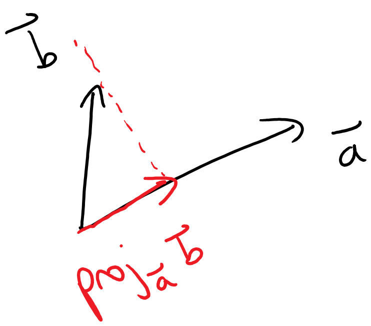
b)



c)



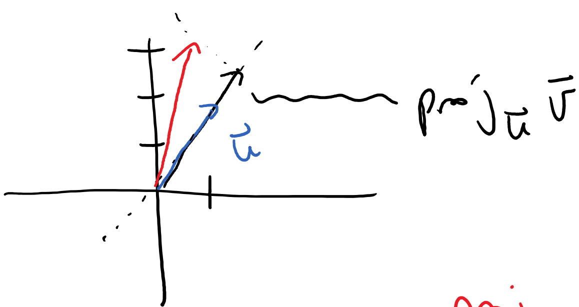
d)



$\text{proj}_{\vec{a}} \vec{b}$?

$$\boxed{\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}}$$

<u>Ex</u>: Find the projection of $\vec{v} = [1, 3]$ onto $\vec{u} = [1, 2]$

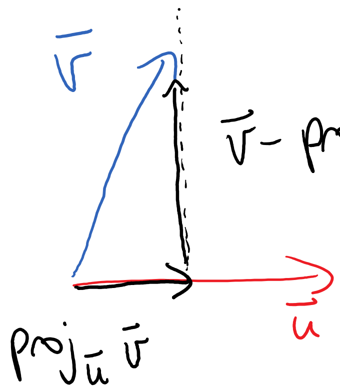


$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}$$

$= \frac{7}{5}[1,2]$

FACT

Given vectors $\bar{u}$ and $\bar{v}$, we can decompose $\bar{v}$ into vectors parallel and perpendicular to $\bar{u}$.



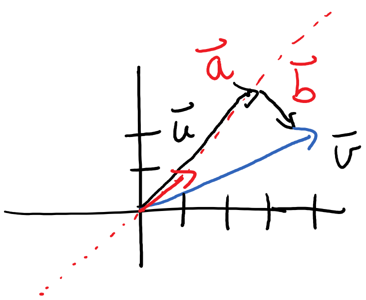
"Orthogonal Decomposition"

$$\boxed{\begin{aligned} \text{proj}_{\bar{u}}\bar{v} + ? &= \bar{v} \\ ? &= \bar{v} - \text{proj}_{\bar{u}}\bar{v} \end{aligned}}$$

<u>Ex</u>: $\bar{u} = [1,1]$ $\quad \bar{v} = [4,2]$

Find $\bar{a}$ and $\bar{b}$ so that $\bar{v} = \bar{a} + \bar{b}$,

$\bar{a}$ is parallel to $\bar{u}$ <u>and</u> $\bar{b}$ is $\perp$ to $\bar{u}$.



$$\vec{a} = \text{proj}_{\bar{u}} \bar{v}$$

$$= \frac{\bar{u} \cdot \bar{v}}{||\bar{u}||^2} \bar{u}$$

$$= \frac{6}{2} [1, 1]$$

$$= [3, 3]$$

$$\bar{a} + \bar{b} = \bar{v}$$

$$\bar{b} = \bar{v} - \vec{a}$$

$$= [4, 2] - [3, 3]$$

$$= [1, -1]$$

No Formula Sheet For Math 251