Week 9 Tuesday

March 3, 2020 8:38 AM

4.2 Determinants Cont'd

Quick Method for 3x3 Determinants

Ex: $A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & -2 & 6 \\ 1 & 0 & 4 \end{bmatrix}$ Find det A

$$\begin{array}{cccccc} + & + & + & - & - & - \\ 1 & 4 & 9 & 1 & 4 & \\ 2 & -2 & 6 & 2 & -2 & \\ 1 & 0 & 4 & 1 & 0 & \\ 18 & 0 & -32 & -8 & 24 & 0 \end{array}$$

$\det A = 18 + 0 - 32 - 8 + 24 + 0 = 2$

Fact

$\boxed{\text{If } A \text{ is upper/lower triangular or diagonal then } |A| = \text{product of diagonal entries}}$

Ex: $\begin{vmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{vmatrix} = 2(-1)(4) = -8$

upper triangular

why?

$\begin{vmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{vmatrix} = 4\begin{vmatrix} 2 & 9 \\ 0 & -1 \end{vmatrix} = 4(2)(-1) = -8$

Row Operations to Calculate |A|

$R_i \leftrightarrow R_j$ changes sign of $|A|$

Can factor a row

$R_i \pm kR_j$ does not change $|A|$

Ex: Calculate $|A|$ using row operations

$$\begin{vmatrix} 1 & -2 & 1 & 9 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 4 & 5 \\ 0 & 1 & 1 & 6 \end{vmatrix}$$

$\boxed{\begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}}$

$$= \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 5 & 1 & -15 \\ 0 & 7 & 1 & -22 \\ 0 & 1 & 1 & 6 \end{vmatrix}$$

$\boxed{R_2 \Leftrightarrow R_4}$

$$= - \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 7 & 1 & -22 \\ 0 & 5 & 1 & -15 \end{vmatrix}$$

$\boxed{\begin{matrix} R_3 - 7R_2 \\ R_4 - 5R_2 \end{matrix}}$

$$= - \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -6 & -64 \\ 0 & 0 & -4 & -45 \end{vmatrix}$$

$\boxed{\frac{R_3}{-6}}$

$$= -(-6) \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{32}{3} \\ 0 & 0 & -4 & -45 \end{vmatrix}$$

$\boxed{R_4 + 4R_3}$

$$= 6 \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{32}{3} \\ 0 & 0 & 0 & -\frac{7}{3} \end{vmatrix}$$

$-45 + 4\left(\frac{32}{3}\right) = -\frac{7}{3}$

upper triangular

$$= 6\left(1 \cdot 1 \cdot 1 \cdot -\frac{7}{3}\right)$$
$$= -14$$

Fact

$\boxed{\text{An } n \times n \text{ matrix } A \text{ is invertible if and only if } \det A \neq 0}$

Properties of det A

1. $\det A^{-1} = \dfrac{1}{\det A}$
2. $\det(AB) = (\det A)(\det B)$
3. $\det(kA) = k^n(\det A)$ where A is $n \times n$
4. $\det(A^T) = \det A$

Ex: Property 3

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$\begin{vmatrix} 3 & 6 \\ 3 & 4 \end{vmatrix} = 3(-2)$$

$$\begin{vmatrix} 3 & 6 \\ 9 & 12 \end{vmatrix} = 3^2(-2)$$

$$\begin{vmatrix} 5a & 5b & 5c \\ 5d & 5e & 5f \\ 5g & 5h & 5i \end{vmatrix} = 5^3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Ex: Show Property 1 by starting with $AA^{-1} = I$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det I$$

$$(\det A)(\det A^{-1}) = 1$$

$$\det A^{-1} = \frac{1}{\det A}$$

Cramer's Rule

Let A be $n \times n$

When $\det A \neq 0$, $A\vec{x} = \vec{b}$ has a unique solution:

$$i^{th} \text{ variable} = \frac{|A_i|}{|A|}$$

where $A_i = A$, with i^{th} column replaced by $\vec{b}$