Week 8 Friday

February 28, 2020 9:45 AM

4.1 Eigenvalues and Eigenvectors

Def

Let A be an $n \times n$ matrix.
If $A\vec{x} = \lambda \vec{x}$ for $\vec{x} \neq \vec{0}$ and some number λ, then
λ is an eigenvalue of A and
$\vec{x}$ " eigenvector of A.

Ex: Show that $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$
$$= 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= 4\vec{x}$$

($\lambda = 4$ is the eigenvalue)

Terminology: $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to $\lambda = 4$

Note: $A\vec{0} = \lambda \vec{0}$ is always true
$\vec{0}$ is not considered an eigenvector

Ex: Find all eigenvectors of $A = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$ corresponding to $\lambda = 6$.

$$A\vec{x} = 6\vec{x}$$

$$A\vec{x} = 6I\vec{x}$$

$$A\vec{x} - 6I\vec{x} = \vec{0}$$

$$(A - 6I)\vec{x} = \vec{0}$$

Solve $[A - 6I \mid \vec{0}\,]$

$$A - 6I = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$$

Solve $\left[\begin{array}{cc|c} -3 & -2 & 0 \\ -3 & -2 & 0 \end{array}\right]$

$\dfrac{R_1}{-3}$ $\left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ -3 & -2 & 0 \end{array}\right]$

$R_2 + 3R_1$ $\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$

$x_2 = t$

$x_1 + \frac{2}{3}x_2 = 0$ $\qquad$ $x_1 = -\frac{2}{3}t$

eigenvectors $\quad \vec{x} = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} t \quad (t \neq 0)$

or $\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} t \qquad (t \neq 0)$

Check: $$A\vec{x} = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ 18 \end{bmatrix} = 6 \begin{bmatrix} -2 \\ 3 \end{bmatrix} \checkmark$$

Fact

To find eigenvectors corresponding to eigenvalue λ: solve $[A - \lambda I \mid \vec{0}]$

Def

The <u>eigenspace</u> E_λ is the set of all eigenvectors of A corresponding to λ, plus the zero vector.

It's a subspace of $\mathbb{R}^n$.

E_λ :

<u>Ex</u>: Find a basis for the eigenspace E_3 for $A = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 0 & 6 \\ 2 & 2 & -1 \end{bmatrix}$

Find eigenvectors : Solve $[A-3I\,|\,\vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -3 & -3 & 6 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right]$$

R_2+3R_1
R_3-2R_1

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

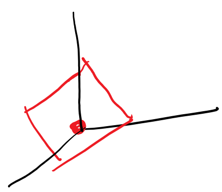
$\boxed{x_2 = s}$ $\boxed{x_3 = t}$

$x_1 + x_2 - 2x_3 = 0$ $\quad \boxed{x_1 = -s + 2t}$

eigenvectors $\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t \quad (\vec{x} \neq \vec{0})$

basis for $E_3 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

eigenspace E_3



Applications

1. Vibrations

natural frequencies = eigenvalues

shapes/directions of vibrations = eigenvectors

② Facial Recognition

$$[\text{Image}] \to \begin{bmatrix} 0 & 18 & 253 \\ & & \cdots \end{bmatrix}$$

eigenvectors = set of "basic" faces

Goal: Express any face as a linear combination of basic faces