Week 8 Thursday

February 27, 2020 10:13 AM

# 3.6 Linear Transformations Cont'd

Ex: $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$

$S : \mathbb{R}^2 \to \mathbb{R}^2$
Rotation by $45°$

Find $[S \circ T]$

$[T] = \begin{bmatrix} & & \end{bmatrix}$ $(2 \times 3)$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ Coefficients

$[S] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta = 45°$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$[S \circ T] = [S][T]$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}$

**Def**

Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$

The <u>inverse</u> of $T$ is a transformation

$T^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ such that :

$$T^{-1}(T(\vec{x})) = \vec{x} \quad \text{and} \quad T(T^{-1}(\vec{x})) = \vec{x}$$

<u>Note</u>: $T^{-1}$ is defined when $[T]$ is invertible.

<u>Ex</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation by $-30^\circ$

Find $[T^{-1}]$

<u>Method I</u> $T^{-1}$ : rotation by $30^\circ$

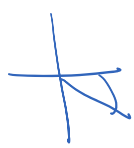
$$[T^{-1}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta = 30^\circ}$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

**FACT** $[T^{-1}] = [T]^{-1}$

" The standard matrix for $T^{-1}$ is the inverse of $[T]$ "

<u>Method II</u>: $[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta = -30^\circ}$



$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Take the inverse

$$\det = \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{-1}{2}\right)$$
$$= 1$$

$$[T^{-1}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

<u>Ex:</u> $T$ is a linear transformation

Given $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Given $T(\vec{v}_1) = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$, $T(\vec{v}_2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $T(\vec{v}_3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

Find $T\begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}$

Recall 2 properties of a linear transformation:

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$$
$$T(c\vec{u}) = c\,T(\vec{u})$$

1) let $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 6 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \text{RREF}$$

$c_1 = 2$
$c_2 = 5$
$c_3 = 1$

2) $T\begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}$

$$= T(2\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3)$$
$$= T(2\vec{v}_1) + T(5\vec{v}_2) + T(\vec{v}_3)$$
$$= 2T(\vec{v}_1) + 5T(\vec{v}_2) + T(\vec{v}_3)$$

T is linear

$$= 2\begin{bmatrix} -5 \\ 8 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ 29 \end{bmatrix}$$

## More details about [T]

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = aT\begin{bmatrix} 1 \\ 0 \end{bmatrix} + bT\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \underset{T\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{(\ )} & \underset{T\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(\ )} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$