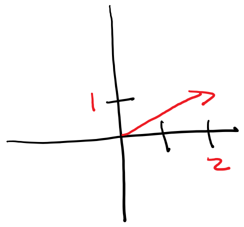
Week 8 Tuesday

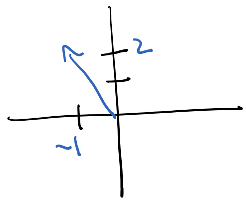
February 25, 2020 8:36 AM

3.6 Linear Transformations

Ex: Transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$

Rotates each vector by 90° counterclockwise





Notation: $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Terminology: $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is the image of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ under T

(output)

Def

$\boxed{\text{The matrix transformation } T_A \text{ multiplies a vector on the left by } A.}$

$T_A(\vec{v}) = A\vec{v}$

Ex: $T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+y \\ x-y \\ 3x+3y \end{bmatrix}$

Find A

$$\begin{bmatrix} & ? & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ \cdots \\ \cdots \end{bmatrix}$$

A must be 3×2

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 3 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ x-y \\ 3x+3y \end{bmatrix}$$

Def

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if:

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

2) $T(c\vec{u}) = c\,T(\vec{u})$ for any real # c

<u>Ex</u>: Show that $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 1+x \end{bmatrix}$ is <u>not</u> linear

- Only need 1 property to fail
- Can use numbers

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) \neq 2\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

<u>Ex</u>: Confirm that T_A satisfies Property 2)

where $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$$T_A(c\vec{u}) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} cu_1 + 2cu_2 \\ cu_1 + cu_2 \end{bmatrix} = c\begin{bmatrix} u_1 + 2u_2 \\ u_1 + u_2 \end{bmatrix} = c\,T_A(\vec{u})$$

FACT

2 Properties

> T is a linear transformation if and only if T is a matrix transformation

T_A

- Tells us which transformations are linear

Def

> The standard matrix for T is the matrix that performs T.

Notation: $[T]$

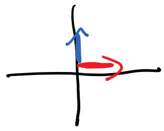
How to Build The Standard Matrix $[T]$

$$[T] = \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) & \cdots & T\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}$$

Works Because T is <u>linear</u>

<u>Ex</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$

Rotates by $90°$



$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$$

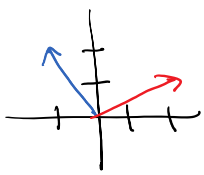
$$[T] = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

$$T\begin{bmatrix}1\\0\end{bmatrix} \qquad T\begin{bmatrix}0\\1\end{bmatrix}$$

Follow-up: $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = [T]\begin{bmatrix}2\\1\end{bmatrix}$

$$= \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}$$

$$= \begin{bmatrix}-1\\2\end{bmatrix}$$

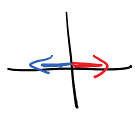


<u>Ex</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$

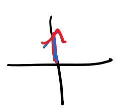
Reflects a vector in y-axis

a) Find $[T]$

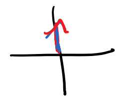
$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$



$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$



$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$



$$[T] = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

b) Find $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

$$= \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$= \begin{bmatrix}-x\\y\end{bmatrix}$$