3.3 The Inverse of a Matrix Cont'd

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Find A^{-2}

$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

$A^{-2} = (A^2)^{-1} = \frac{1}{4}\begin{bmatrix} 22 & -10 \\ -15 & 7 \end{bmatrix}$

Ex: A, B, X are all $n \times n$ invertible matrices.
Solve for X given $(AX)^{-1} = BA$

$((AX)^{-1})^{-1} = (BA)^{-1}$

$AX = (BA)^{-1}$

$AX = A^{-1}B^{-1}$

Multiply by A^{-1} on the left

$A^{-1}AX = A^{-1}A^{-1}B^{-1}$

$X = A^{-2}B^{-1}$

Elementary Matrices: represent row operations

$\boxed{\text{How is } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ changing?}}$

Ex: a) $E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ represents $3R_1$ $\begin{bmatrix} \quad \end{bmatrix}$

$\frac{R_1}{3}$ $\begin{bmatrix} \quad \end{bmatrix}$ undoes it

$E_1^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

b) $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents $R_1 \leftrightarrow R_2$ $\begin{bmatrix} \quad \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} \quad \end{bmatrix}$ undoes it

$E_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) $E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ represents $R_2 + 2R_1$ $\begin{bmatrix} \quad \end{bmatrix}$

$R_2 - 2R_1$ $\begin{bmatrix} \quad \end{bmatrix}$ undoes it

$\boxed{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

FACT

Elementary matrices act on the <u>left</u> of A

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$

$$\cancel{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2a & \\ 2c & \end{bmatrix}}$$

<u>Ex:</u> $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

Write A and A^{-1} as a product of elementary matrices.

$\frac{R_1}{2}$ $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $\qquad E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$R_1 - \frac{1}{2}R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad E_2 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$

$$\underbrace{E_2 E_1}A = I$$

$$A^{-1} = E_2 E_1$$

$$A = (A^{-1})^{-1}$$

$$A = E_1^{-1} E_2^{-1}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$2R_1 \begin{bmatrix} & \\ & \end{bmatrix}$ $\quad R_1 + \frac{1}{2}R_2 \begin{bmatrix} & \\ & \end{bmatrix}$

<u>Ex:</u> $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

Write A and A^{-1} as a product of

elementary matrices.

$$\frac{R_1}{2}\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix} \qquad E_1 = \begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}$$

$$R_2 - R_1 \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix} \qquad E_2 = \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$$

$$\frac{R_2}{-1}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} \qquad E_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$R_1 - 2R_2 \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad E_4 = \begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$$

How does $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ change?

$$\underbrace{E_4 E_3 E_2 E_1}_{A^{-1} = E_4 E_3 E_2 E_1} A = I$$

$$\begin{aligned} A &= (A^{-1})^{-1} \\ &= (E_4 E_3 E_2 E_1)^{-1} \\ &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} \\ &= \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\end{aligned}$$

$\boxed{2R_1 \; [\quad]}$ $\qquad$ $\boxed{R_2 + R_1 \; [\quad]}$