Week 5 Thursday
February 6, 2020 9:42 AM

3.2 Matrix Algebra Cont'd

Def
A and B **commute** if $AB = BA$

Ex: Do A and B commute?

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 11 & 39 \\ 13 & 37 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 39 \\ 13 & 37 \end{bmatrix}$$

$\boxed{\text{Yes}}$

6 Properties of Matrices

$$\text{①} \quad (AB)C = A(BC)$$

Ex:

$$(AB)C = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -10 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -60 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 12 \\ -4 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -60 \end{bmatrix} \quad \checkmark$$

$$\text{②} \quad A(B+C) = AB + AC$$

Ex:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 50 \end{bmatrix}$$

$$AB+AC = \begin{bmatrix}17\\39\end{bmatrix} + \begin{bmatrix}5\\11\end{bmatrix} = \begin{bmatrix}22\\56\end{bmatrix} \checkmark$$

$$\boxed{\begin{array}{ll} ③ & AI = A \\ ④ & IA = A \end{array}}$$

Section 3.1

Ex:

$$\begin{bmatrix}6 & 5\\4 & 3\\2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}6 & 5\\4 & 3\\2 & 1\end{bmatrix} \checkmark$$

$$\boxed{⑤ \quad (A \pm B)^T = A^T \pm B^T}$$

means $\begin{cases} (A+B)^T = A^T + B^T \\ (A-B)^T = A^T - B^T \end{cases}$

Ex: $A = \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}$ $B = \begin{bmatrix}1 & 4\\1 & 2\end{bmatrix}$

$$(A+B)^T = \begin{bmatrix}2 & 6\\2 & 5\end{bmatrix}^T = \begin{bmatrix}2 & 2\\6 & 5\end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix} + \begin{bmatrix}1 & 1\\4 & 2\end{bmatrix} = \begin{bmatrix}2 & 2\\6 & 5\end{bmatrix} \checkmark$$

$$\boxed{⑥ \quad (A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$

$$(AB)^T = \begin{bmatrix} 3 & 8 \\ 4 & 10 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 \\ 8 & 10 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 8 & 10 \end{bmatrix} \checkmark$$

How are these properties established?

Ex: Show that $A+B = B+A$

Plan: Show that $[A+B]_{ij} = [B+A]_{ij}$

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij}$$

real numbers

$$= [B]_{ij} + [A]_{ij}$$

$$= [B+A]_{ij}$$

Ex: Expand $(A+B)^2$ and simplify

$$(A+B)^2 = (A+B)(A+B)$$

$$= A^2 + AB + BA + B^2$$

$\underline{\text{Ex}}$: Show that $A^T A$ is symmetric

$\boxed{M \text{ is symmetric}: \quad M^T = M}$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Show that $(A^T A)^T = A^T A$

$$(A^T A)^T = A^T (A^T)^T \quad \text{(Property 6)}$$
$$= A^T A \checkmark$$

3.3 The Inverse of a Matrix

Def

An $n \times n$ matrix A is <u>invertible</u> if there exists a matrix A^{-1} (also $n \times n$) so that $AA^{-1} = I$ and $A^{-1}A = I$

A^{-1} is called the <u>inverse</u> of A

$\underline{\text{Ex}}$: Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

Check that $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

$A^{-1}A = I$ ✓

Notes:

1) Not every square matrix has an inverse

2) Only need to check 1 of:

$$AA^{-1} = I \quad \text{or} \quad A^{-1}A = I$$