Week 10 Friday

March 13, 2020 10:20 AM

4.4 Diagonalization Cont'd

Ex: Diagonalize $A = \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$ (if possible)

$\lambda = 4, 4$ (A is lower triangular)

E_4 : $[A - 4I \mid \vec{0}]$

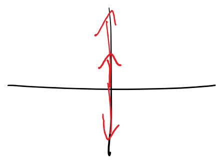
$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$\boxed{x_2 = t}$ $\boxed{x_1 = 0}$

eigenvectors $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$

basis for $E_4 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$



Can't build P

Can't diagonalize A

DETAILS

Characteristic polynomial of $A = |A - \lambda I|$

$$= \begin{vmatrix} 4-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix}$$

$$= (4-\lambda)^2$$

Algebraic multiplicity of $\lambda = 4$ is 2

Geometric " " is 1

Geo. Mult. < Alg. Mult. $\Rightarrow$ Can't diagonalize A

Ex: $\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}^2 = \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 16 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (-4)^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

FACT
If D is diagonal then D^n is diagonal, with n^{th} powers on the diagonal.

FACT
If $P^{-1}AP = D$ then $A^n = PD^nP^{-1}$

Why?

$$P^{-1}AP = D$$
$$\cancel{PP^{-1}}AP = PD$$
$$A\cancel{PP^{-1}} = PDP^{-1}$$
$$A^n = (PD\cancel{P^{-1})(P}D\cancel{P^{-1})} \cdots (\cancel{P}DP^{-1})$$
$$A^n = PD^nP^{-1}$$

Ex: $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ diagonalizes A

to produce $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Find A^k

$$P^{-1}AP = D$$
$$\cancel{PP^{-1}}A\cancel{PP^{-1}} = PDP^{-1}$$
$$A^k = (PD\cancel{P^{-1})(P}D\cancel{P^{-1})} \cdots (\cancel{P}DP^{-1})$$
$$A^k = PD^kP^{-1}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{[P \mid I] \rightsquigarrow [I \mid P^{-1}]}$$

$$= \begin{bmatrix} 3^k & 0 & 4^k \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3^k & 0 & 4^k - 3^k \\ 0 & 3^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \Leftarrow \; -3^k + 4^k$$

Ex: Application of A^n

Consider a company with 1,000 machines

$W =$ a machine is working

$B =$ " broken

Probability Matrix

$$\begin{array}{c|c} & \begin{array}{cc} W & B \end{array} \leftarrow \text{today} \\ \begin{array}{c} W \\ B \end{array} & \begin{bmatrix} 0.99 & 0.5 \\ 0.01 & 0.5 \end{bmatrix} \end{array} = A$$

$\uparrow$ tomorrow

State Vector

$$\vec{x} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix} \begin{array}{c} W \\ B \end{array}$$

All machines are working today

$$A\vec{x} = \begin{bmatrix} 0.99 & 0.5 \\ 0.01 & 0.5 \end{bmatrix} \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 990 \\ 10 \end{bmatrix} \end{array} \begin{array}{l} W \\ B \end{array}$$

Tomorrow

$$A^2\vec{x} = \begin{bmatrix} 0.99 & 0.5 \\ 0.01 & 0.5 \end{bmatrix} \begin{bmatrix} 990 \\ 10 \end{bmatrix} \approx \begin{bmatrix} 985 \\ 15 \end{bmatrix} \begin{array}{l} W \\ B \end{array}$$

2 days from now

As $n \to \infty$, $\quad A^n\vec{x} \longrightarrow \begin{bmatrix} 980 \\ 20 \end{bmatrix} \begin{array}{l} W \\ B \end{array}$

In the long run, 20 will be broken each day.