# Week 10 Thursday

March 12, 2020 9:39 AM

## 4.3 Finding Eigenvalues Cont'd

**Warm-Up:** $A\begin{bmatrix}-1\\5\end{bmatrix} = 2\begin{bmatrix}-1\\5\end{bmatrix}$

Find $A^4\begin{bmatrix}-1\\5\end{bmatrix}$

$= \lambda^4\begin{bmatrix}-1\\5\end{bmatrix}$

$= 16\begin{bmatrix}-1\\5\end{bmatrix}$

$= \begin{bmatrix}-16\\80\end{bmatrix}$

Recap Fact 5 from yesterday

**Ex:** $A$ has eigenvalue $\lambda_1 = -2$ corresponding to $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$
and " $\lambda_2 = 3$ " $\vec{v}_2 = \begin{bmatrix}2\\-1\end{bmatrix}$

Find $A^3\begin{bmatrix}11\\2\end{bmatrix}$

1. let $c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}11\\2\end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & -1 & 2 \end{array}\right]$$

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

$\begin{bmatrix}11\\2\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix} + 4\begin{bmatrix}2\\-1\end{bmatrix}$

(2) $A^3\begin{bmatrix}11\\2\end{bmatrix} = A^3\left(3\begin{bmatrix}1\\2\end{bmatrix}+4\begin{bmatrix}2\\-1\end{bmatrix}\right)$

$$= c_1\lambda_1^3\vec{v}_1 + c_2\lambda_2^3\vec{v}_2$$

$$= 3(-2)^3\begin{bmatrix}1\\2\end{bmatrix} + 4(3)^3\begin{bmatrix}2\\-1\end{bmatrix}$$

$$= -24\begin{bmatrix}1\\2\end{bmatrix} + 108\begin{bmatrix}2\\-1\end{bmatrix}$$

$$= \begin{bmatrix}192\\-156\end{bmatrix}$$

## 4.4 Diagonalization

Def

An $n \times n$ matrix $A$ is diagonalizable if there exist an invertible matrix $P$ and a diagonal matrix $D$ so that

$$P^{-1}AP = D$$

Ex: Find $P$ that diagonalizes $A = \begin{bmatrix}2 & 0 & -2\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}$

→ Find a basis for each eigenspace of $A$

$\lambda = 2, 3, 3$

($A$ is upper triangular ⇒ eigenvalues are the diagonal entries)

$E_2$ : Solve $[A-2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$x_1 = t$

$x_2 = 0$
$x_3 = 0$

eigenvectors $\quad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t \quad (t \neq 0)$

basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$E_3$: $\quad$ Solve $[\,A - 3I \mid \vec{0}\,]$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$\boxed{x_2 = s}$ $\quad \boxed{x_3 = t}$

$x_1 + 2x_3 = 0 \quad \boxed{x_1 = -2t}$

eigenvectors $\quad \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$

basis for $E_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

To build P: put basis vectors into columns

To build D: eigenvalues on the diagonal, in the same order as columns of P

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To check: Find $P^{-1}$ by $[P \mid I] \rightsquigarrow [I \mid P^{-1}]$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check that $P^{-1}AP = D$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \checkmark$$

$P^{-1}$ will always exist because columns of P are linearly independent (Section 4.3)

FACT

A is diagonalizable if and only if geometric multiplicity = algebraic multiplicity for all eigenvalues.

# basis vectors in eigenspace

in previous example
$\lambda = 2, 3, 3$
alg. mult. of $\lambda = 3$ is 2

Intuitively: A is diagonalizable exactly when you have enough columns to make P square.