Week 10 Wednesday

March 11, 2020 9:06 AM

4.3 Finding Eigenvalues Cont'd

Ex: A is 5×5

$|A-\lambda I| = (7-\lambda)^3(9-\lambda)^2$

Basis for $E_7 = \{[\ ]\}$

Basis for $E_9 = \{[\ ],[\ ]\}$

Eigenvalue	Algebraic Multiplicity	Geometric Mult.
7	3	1
9	2	2

FACT
For each eigenvalue, geometric multiplicity $\leq$ algebraic multiplicity

When geo. mult. = alg. mult for each eigenvalue, then A has a useful property. (Section 4.4)

5 Facts about Eigenvalues

1. A is invertible if and only if 0 is <u>not</u> an eigenvalue of A

A is invertible

$\iff \det A \neq 0$

$\iff \det(A - 0I) \neq 0$

$\iff 0$ is not an eigenvalue of A

$\textcircled{2}$ If A is invertible and $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

$$A\vec{x} = \lambda\vec{x}$$

Multiply by A^{-1} : $\quad A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$

$$\vec{x} = A^{-1}\lambda\vec{x}$$

$$\vec{x} = \lambda A^{-1}\vec{x} \qquad (\lambda \text{ is a constant})$$

$$\frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$$

$$A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$

eigenvector eigenvalue

$\textcircled{3}$ If $A\vec{x} = \lambda\vec{x}$ then $A^n\vec{x} = \lambda^n\vec{x}$ (for $n = 2, 3, \dots$)

$$\begin{aligned} A^n\vec{x} &= A^{n-1}A\vec{x} \\ &= A^{n-1}\lambda\vec{x} \\ &= \lambda A^{n-1}\vec{x} \qquad (\lambda \text{ is a constant}) \\ &= \lambda^2 A^{n-2}\vec{x} \\ &\;\vdots \\ &= \lambda^n\vec{x} \end{aligned}$$

$\textcircled{4}$ If $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of $A + kI$ with eigenvalue $\lambda + k$

$$(A+kI)\vec{x} = A\vec{x} + kI\vec{x}$$
$$= \lambda\vec{x} + k\vec{x}$$
$$= \underbrace{(\lambda+k)}_{\text{eigenvalue}}\ \vec{x}$$

eigenvector

<u>Ex</u>: Summary of Facts 2-4

A is invertible with eigenvalues -2 and 3

Eigenvalues of A^{-1} : $-\frac{1}{2}$, $\frac{1}{3}$

A^4 : $(-2)^4$, 3^4

$A-5I$: -7, -2

$$\boxed{\begin{array}{l} \text{⑤ Suppose } A \text{ has eigenvectors } \vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \\ \quad \text{corresponding to } \lambda_1, \lambda_2, \dots, \lambda_m . \\ A^k(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m) = c_1\lambda_1^k\vec{v}_1 + \dots + c_m\lambda_m^k\vec{v}_m \end{array}}$$

- Generalization of Fact 3
- Coefficients are preserved

<u>Ex</u>: A is a 2x2 matrix with

eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ corresponding to $\lambda_1 = 2$

" $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ " $\lambda_2 = 3$

Find $A^4\begin{bmatrix} -1 \\ 12 \end{bmatrix}$

1) $$\begin{bmatrix} -1 \\ 12 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & 2 & 12 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

$$\begin{bmatrix} -1 \\ 12 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

2) $$A^4 \begin{bmatrix} -1 \\ 12 \end{bmatrix} = A^4 \left(2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right)$$

$$= 2 \lambda_1^4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \lambda_2^4 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= 2 (2)^4 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 (3)^4 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= 32 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 243 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -211 \\ 582 \end{bmatrix}$$