Week 10 Tuesday

March 10, 2020 8:32 AM

Test 3
Thurs Mar 26^{th}

4.3 Finding Eigenvalues

Ex: Find all eigenvalues of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$

Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -4-\lambda & 3 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-4-\lambda)(7-\lambda) = 0$$

$$\lambda = 1, -4, 7$$

FACT

If A is diagonal or upper/lower triangular then eigenvalues of A are the diagonal entries.

Ex: Find all eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{bmatrix} + & - & + \\ & & \\ & & \end{bmatrix}$$

$$(3-\lambda)\begin{vmatrix} -5-\lambda & 1 \\ -6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & 1 \\ 6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & -5-\lambda \\ 6 & -6 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-5-\lambda)(2-\lambda)+6\right] + \left[7(2-\lambda)-6\right] + \left[-42-6(-5-\lambda)\right] = 0$$

$$(3-\lambda)\left[\lambda^2+3\lambda-4\right] + \left[-7\lambda+8\right] + \left[6\lambda-12\right] = 0$$

$$\begin{array}{rrrr} & 3\lambda^2 & +9\lambda & -12 \\ -\lambda^3 & -3\lambda^2 & +4\lambda & \\ & & -7\lambda & +8 \\ + & & 6\lambda & -12 \\ \hline -\lambda^3 & & +12\lambda & -16 \end{array}$$

$$-\lambda^3+12\lambda-16=0$$

$$\lambda^3-12\lambda+16=0$$

Integer Roots Theorem

If a polynomial has integer coefficients and the leading coefficient is 1 then any integer roots divide the constant.

Possible roots (solutions) : $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

Check $\lambda=-1$: $(-1)^3-12(-1)+16=0$? No

$\lambda=1$ No

$\lambda=2$ $2^3-12(2)+16=0$? YES

$\lambda = 2$ is a solution $\Rightarrow$ $(\lambda-2)$ is a factor

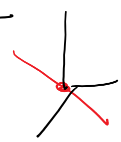
Long Division

$$
\begin{array}{r}
\lambda^2 + 2\lambda - 8 \\
(\lambda-2) \overline{\big)\ \lambda^3 + 0\lambda^2 - 12\lambda + 16} \\
-(\lambda^3 - 2\lambda^2) \qquad\qquad \\
\hline
2\lambda^2 - 12\lambda + 16 \\
-(2\lambda^2 - 4\lambda) \qquad \\
\hline
-8\lambda + 16 \\
-(-8\lambda + 16) \\
\hline
0
\end{array}
$$

$$\lambda^3 - 12\lambda + 16 = 0$$
$$(\lambda-2)(\lambda^2+2\lambda-8) = 0$$
$$(\lambda-2)(\lambda-2)(\lambda+4) = 0$$
$$(\lambda-2)^2(\lambda+4) = 0$$
$$\lambda = 2, -4$$

Given: Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Means $A\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$



Basis for $E_{-4} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

FACT

$$\boxed{\text{When bases for different eigenspaces are combined, the new set is linearly independent}}$$

e.g. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent

Useful in Section 4.4

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Def

- $\underline{\text{Characteristic equation of } A}$: $|A-\lambda I| = 0$
- $\underline{\text{Algebraic multiplicity of an eigenvalue } \lambda_i}$:
  exponent on $(\lambda - \lambda_i)$ in characteristic equation
- $\underline{\text{Geometric multiplicity of an eigenvalue}}$:
  # of basis vectors in the eigenspace

Ex: A has characteristic equation

$$\lambda^3 - 12\lambda + 16 = 0.$$

Given

$$E_2 = \text{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$$

Solve $[A-2I \mid \vec{0}]$

$\vec{x} = \begin{bmatrix}1\\1\\0\end{bmatrix} t$

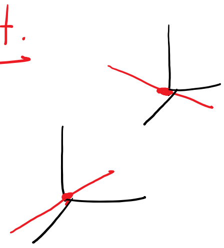
$$E_{-4} = \text{span}\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)$$

Find algebraic and geometric multiplicities for all eigenvalues.

$$\lambda^3 - 12\lambda + 16 = 0$$

$$\rightarrow (\lambda-2)^2(\lambda+4) = 0$$

	Alg. Mult.	Geo. Mult.
$\lambda = 2$	2	1
$\lambda = -4$	1	1



$$\boxed{\text{geo. mult.} \le \text{alg. mult.}}$$