3.6 Linear Transformations Cont'd

Def

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if:

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
for any $\vec{u}, \vec{v}$ in $\mathbb{R}^n$

AND

2) $T(c\vec{u}) = c\,T(\vec{u})$
c: any real #

<u>Ex</u>: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1+x \end{bmatrix}$

Show that T is <u>not</u> linear

<u>Note</u>: only need 1 property to fail
Can use numbers

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq 2T\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

<u>Ex</u>: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

Confirm that T_A satisfies Property 2)

<u>Recall</u>: $T_A(\vec{x}) = A\vec{x}$

Show $T_A(c\vec{u}) = c\,T_A(\vec{u})$

$$
\begin{aligned}
T_A(c\vec{u}) &= A(c\vec{u}) \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} \\
&= \begin{bmatrix} cu_1 + cu_2 \\ cu_1 + 2cu_2 \\ cu_1 + cu_2 \end{bmatrix} \\
&= c \begin{bmatrix} u_1 + u_2 \\ u_1 + 2u_2 \\ u_1 + u_2 \end{bmatrix} \\
&= c \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= c\,T_A(\vec{u}) \quad \checkmark
\end{aligned}
$$

FACT

T is a linear transformation if and only if T is a matrix transformation.

has 2 properties

T can be described by a matrix

<u>Interpretation</u>: A transformation T can be described by a matrix exactly when T has the 2 linear properties.

Def

The <u>standard matrix</u> for T is the matrix that performs T.

Notation: $[T]$

How to find the standard matrix $[T]$:

$$[T] = \begin{array}{c} \begin{bmatrix} \Big(\quad\Big) & \Big(\quad\Big) \end{bmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ T\begin{bmatrix}1\\0\end{bmatrix} & T\begin{bmatrix}0\\1\end{bmatrix} \end{array} \end{array}$$

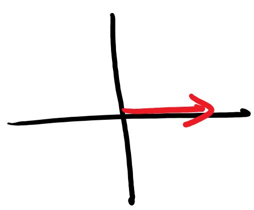
Why this works: T is linear

In higher dimensions:

$$[T] = \begin{bmatrix} T\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} & T\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} & \cdots & T\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} \end{bmatrix}$$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$ reflects a vector in the y-axis.

a) Find $[T]$



$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

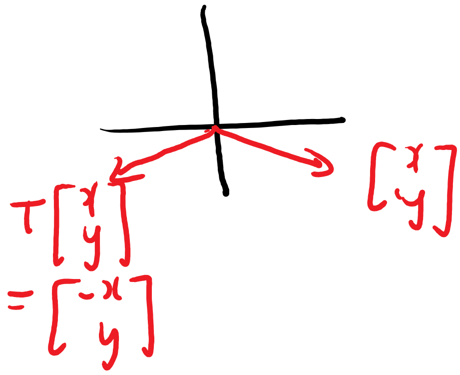
$$[T] = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

b) Find $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$

$$= [T]\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

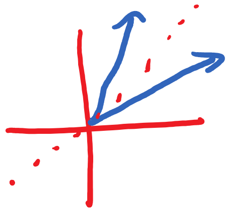
$$= \begin{bmatrix} -x \\ y \end{bmatrix}$$



Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

reflects a vector in the line $y = x$

Find $[T]$



$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

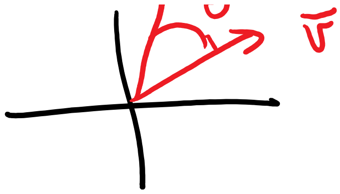
$$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

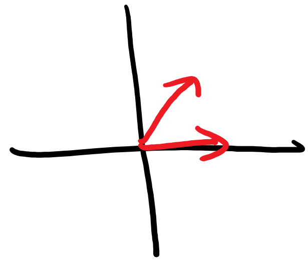
Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

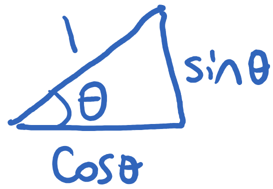
rotation by angle θ (counterclockwise)

Find $[T]$

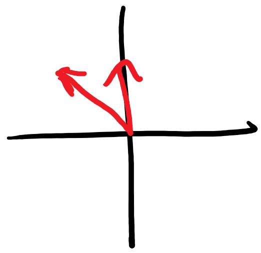


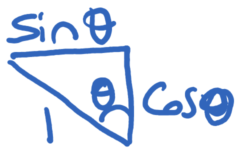
$$T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$





$$T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$





$$[T]=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Valid for any θ ☺

⊛ Know this

Ex: Rotate $\begin{bmatrix}1\\1\end{bmatrix}$ by $30°$ clockwise

$$[T]=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \quad \theta=-30°$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3}+1 \\ -1+\sqrt{3} \end{bmatrix}$$