3.3 Matrix Inverses Cont'd

RECALL If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of A

is $|A|$ or $\det A = ad - bc$

$$\boxed{A^{-1} = \begin{cases} \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, & \text{if } \det A \neq 0 \\ \text{undefined}, & \text{if } \det A = 0 \end{cases}}$$

Ex: Find A^{-1} (if it exists)

a) $A = \begin{bmatrix} 1 & -4 \\ -7 & 2 \end{bmatrix}$

$\det A = 1(2) - (-4)(-7)$
$= -26$

$A^{-1} = \frac{-1}{26}\begin{bmatrix} 2 & 4 \\ 7 & 1 \end{bmatrix}$

Check: $AA^{-1} = I$ ✓

b) $A = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$

$\det A = 0$

A^{-1} does not exist

Why The Formula Works

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$ ✓

Recall: System of equations can be written $\underbrace{A}_{\text{Coefficients}} \underbrace{\vec{x}}_{\text{variables}} = \underbrace{\vec{b}}_{\text{Constants}}$

If A^{-1} exists:

$$A\vec{x} = \vec{b}$$

$$\underbrace{A^{-1}A}\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

FACT

If A^{-1} exists, then the system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$

Ex: Use A^{-1} to solve

$$\begin{cases} 4x - 5y = -6 \\ -5x + 6y = 7 \end{cases}$$

$$A = \begin{bmatrix} 4 & -5 \\ -5 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -6 \\ 7 \end{bmatrix}$$

$$= -\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad x=1,\ y=2$$

Finding A^{-1} for $n \times n$ matrices

$$[A \mid I] \xrightarrow[\text{Gauss-Jordan}]{} [I \mid A^{-1}]$$

Ex: Find A^{-1} for $A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & -2 & -2 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -8 & 2 & -6 & 1 \end{array}\right]$$

$$R_3/(-8)\left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{-2}{8} & \frac{6}{8} & \frac{-1}{8} \end{array}\right]$$

$$\begin{array}{l} R_1-8R_3 \\ R_2+3R_3 \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & \frac{2}{8} & \frac{2}{8} & \frac{-3}{8} \\ 0 & 0 & 1 & \frac{-2}{8} & \frac{6}{8} & \frac{-1}{8} \end{array}\right] \qquad \boxed{1+3\left(\frac{-2}{8}\right)} \qquad \boxed{-2+3\left(\frac{6}{8}\right)}$$

$$[\,I \mid A^{-1}\,]$$

$$A^{-1} = \frac{1}{8}\begin{bmatrix} 0 & -8 & 8 \\ 2 & 2 & -3 \\ -2 & 6 & -1 \end{bmatrix}$$

Ex: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 6 \\ 1 & 1 & 5 \end{bmatrix}$

$$[\,A \mid I\,] \rightsquigarrow [\,I \mid A^{-1}\,]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 6 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2-R_1 \\ R_3-R_1 \end{array}\left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 0 & & & \end{array}\right]$$

If a zero row appears on the left side, then A^{-1} does not exist.

Terminology

"A^{-1} exists" means the same thing as "A is invertible"