Test 2 Fri Oct 18^{th} (6 Questions)
2.3, 2.4, 3.1-3.3

3.2 Matrix Algebra Cont'd

Def
A and B <u>Commute</u> if $AB = BA$

(Recall $AB \neq BA$ for most matrices)

<u>Ex</u>: Do A and B Commute?

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 39 \\ 13 & 37 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 9 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 39 \\ 13 & 37 \end{bmatrix}$$

Yes

<u>6 Properties of Matrices</u>

① $(AB)C = A(BC)$

<u>Ex:</u> $\underbrace{\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix}}\begin{bmatrix} 1 & 6 \end{bmatrix}$

$= -10\begin{bmatrix} 1 & 6 \end{bmatrix}$

$= \begin{bmatrix} -10 & -60 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \end{bmatrix}\underbrace{\begin{bmatrix} 2 \\ -4 \end{bmatrix}\begin{bmatrix} 1 & 6 \end{bmatrix}}$

$= \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 12 \\ -4 & -24 \end{bmatrix}$

$= \begin{bmatrix} -10 & -60 \end{bmatrix}$ ✓

② $A(B+C) = AB + AC$

<u>Ex:</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 50 \end{bmatrix}$

$AB + AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 17 \\ 39 \end{bmatrix} + \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$= \begin{bmatrix} 22 \\ 50 \end{bmatrix}$ ✓

$\left[\ 50\ \right]$ ✓

③ ④ $IA = A$

$AI = A$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

⑤ $(A \pm B)^T = A^T \pm B^T$

Means $\begin{cases} (A+B)^T = A^T + B^T \\ (A-B)^T = A^T - B^T \end{cases}$

⑥ $(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$

COUNTER-INTUITIVE

Ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$

$(AB)^T = \left(\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} \right)^T = \begin{bmatrix} 3 & 8 \\ 4 & 10 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 \\ 8 & 10 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 8 & 10 \end{bmatrix}$ ✓

Ex: Expand and simplify: $(A+B)^2$

$(A+B)^2 = (A+B)(A+B)$

$= AA + AB + BA + BB$

$$= A^2 + AB + BA + B^2$$

Ex: Show that A^TA is symmetric

Recall: M is symmetric if $M^T = M$

Need to show that $(A^TA)^T = A^TA$

Start with the more complicated side:

$$(A^TA)^T = A^T(A^T)^T$$

$$= A^TA \checkmark$$

Property 6
$(CB)^T = B^TC^T$

3.3 The Inverse of a Matrix

Def
An $n \times n$ matrix A is invertible if there exists a matrix A^{-1} (also $n \times n$) so that $AA^{-1} = I$ and $A^{-1}A = I$

A^{-1} is called "the inverse of A"

Ex: Let $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

Check that $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$\rightarrow$ Show that $AA^{-1} = I$ and $A^{-1}A = I$

$$AA^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$A^{-1}A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Notes:

1) $AA^{-1} = I$ if and only if $A^{-1}A = I$

   Only need to check 1 of these

2) Not every square matrix is invertible

   Invertible $\Rightarrow$ Square

   Square $\not\Rightarrow$ Invertible

$$AA^{-1} = I$$

Analogy with $7\left(\frac{1}{7}\right) = 1$

"A^{-1} is undoing A"