Week 12 Monday
November 18, 2019 7:46 AM

Test Fri Nov 29
4.3, 4.4, 5.1-5.4 (5 Questions)
Practice Problems on Website

Omit Sugg HW Section 5.3 # 15, 17, 21

5.4 Orthogonal Diagonalization

Recall: If Q is an orthogonal matrix then $Q^{-1} = Q^T$

Def
An $n\times n$ matrix A is <u>orthogonally diagonalizable</u> if there exists an orthogonal matrix Q so that $Q^T A Q = D$

diagonal

Compare to Section 4.4: $P^{-1}AP = D$

Ex: $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ has $\lambda = 4, 7$

Find Q that orthogonally diagonalizes A

→ Section 4.4 Basis for each eigenspace

Orthogonal diagonalization:
<u>orthonormal</u> basis for each eigenspace

E_4: $[A-4I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-R_1 \\ R_3-R_1 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$x_2 = s$ $\quad x_3 = t$

$x_1 + x_2 + x_3 = 0 \rightarrow x_1 = -s - t$

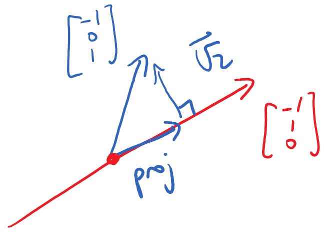
$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Use Gram-Schmidt to get an <u>orthonormal</u> basis

partial basis $X = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \text{proj}_X \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Scale

$$2\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Orthogonal basis $= \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

Orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Eigenspace E_7 :

$$[A - 7I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$\rightsquigarrow \quad \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_3 = t$

$$x_1 - x_3 = 0 \rightarrow x_1 = t$$

$$x_2 - x_3 = 0 \rightarrow x_2 = t$$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$$

Ortho<u>normal</u> basis $= \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Q: Put orthonormal eigenvectors into the columns

$$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

Follow-up :

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

With $Q^T A Q = D$, easy to compute A^{50} etc.

Compute A^{50} etc.

Spectral Theorem

An $n \times n$ matrix A can be orthogonally diagonalized if and only if A is symmetric.

$A^T = A$

Ex: Can A be orthogonally diagonalized?

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ No

b) $A = \begin{bmatrix} 0 & -8 \\ -8 & -5 \end{bmatrix}$ Yes

Recall: Outer Product Expansion (Section 3.1)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

weird

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6 \\ 15 & 18 \end{bmatrix} + \begin{bmatrix} 14 & 16 \\ 28 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \checkmark$$

FACT: "Spectral Decomposition"

Let A be a symmetric matrix.

Given orthonormal eigenvectors $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$

corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

The <u>spectral decomposition</u> of A is

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \cdots + \lambda_n \vec{q}_n \vec{q}_n^{\,T}$$

Allows us to find a matrix with specific eigenvalues/vectors.

<u>Note</u>: Vectors $\vec{q}$ are <u>column</u> vectors

must be <u>normalized</u> (length 1)