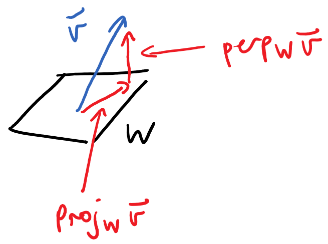
5.2 Complements and Projections Cont'd

Def

The <u>orthogonal decomposition</u> of $\vec{v}$ with respect to subspace W is:

$$\vec{v} = \underbrace{\text{proj}_W \vec{v}}_{\text{in } W} + \underbrace{\text{perp}_W \vec{v}}_{\text{in } W^\perp}$$



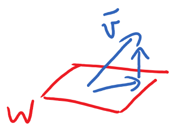
Recall $\text{proj}_{\vec{u}} \vec{z} = \dfrac{\vec{u} \cdot \vec{z}}{\|\vec{u}\|^2} \vec{u}$

Ex: W has orthogonal basis

$$\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}}_{\vec{w}_2} \right\}$$

Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

Find the orthogonal decomposition of $\vec{v}$ with respect to W.



$$\text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v}$$

(if you have an <u>orthogonal</u> basis)

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\|\vec{w}_1\|^2} \vec{w}_1 + \ldots$$

$$= \frac{-4}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{11}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{-18}{9} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{11}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 4 \\ -11 \\ 40 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4 \\ -11 \\ 40 \end{bmatrix}$$

$\text{perp}_W \vec{v}$:

$$\vec{v} = \text{proj}_W \vec{v} + \text{perp}_W \vec{v}$$

$$\text{perp}_W \vec{v} = \vec{v} - \text{proj}_W \vec{v}$$

$$= \frac{9}{9}\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 4 \\ -11 \\ 40 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 5 \\ 20 \\ 5 \end{bmatrix}$$

$$\vec{v} = \underset{(\text{in } W)}{\text{proj}_W \vec{v}} + \underset{(\text{in } W^\perp)}{\text{perp}_W \vec{v}}$$

5.3 Gram-Schmidt Orthogonalization

How to get an orthogonal basis

Ex: Let $W = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} \right)$

Find an orthogonal basis for W.

Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

($\vec{v}_2$ is the component of $\begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$ that is orthogonal to X)

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix} - \frac{\cancel{12}^{\,3}}{\cancel{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

orthogonal ✓

$$\vec{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} - \text{proj}_{\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -3 \\ -4 \\ -2 \end{bmatrix} \underbrace{\cancel{- \frac{(-8)}{4}}}_{+2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-7)}{10} \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

Can scale to eliminate fractions

$$10\vec{v}_3 = \begin{bmatrix} 10 \\ -30 \\ -40 \\ -20 \end{bmatrix} + \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \end{bmatrix} + 7 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ -17 \\ -13 \\ 14 \end{bmatrix}$$

$$X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 16 \\ -17 \\ -13 \\ 14 \end{bmatrix} \right\}$$

orthogonal ✓

If we wanted an <u>orthonormal</u> basis :

$$\left\{ \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}, \frac{1}{\sqrt{910}}\begin{bmatrix}16\\-17\\-13\\14\end{bmatrix} \right\}$$

This whole process is called
"Gram-Schmidt Orthogonalization"