Week 11 Tuesday

November 12, 2019 9:41 AM

Test Average = 80%

5.1 Orthogonality Cont'd

Recap

orthogonal set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$

orthonormal set $\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$

orthogonal matrix Q :
has orthonormal columns

!

Fact
A matrix Q is orthogonal if and only if
$$Q^T Q = I$$

$$\begin{bmatrix} \vec{v}_1 \\ \\ \\ \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & \\ & & 1 \end{bmatrix}$$

Fact
If Q is orthogonal then $Q^{-1} = Q^T$

Why?
Q is orthogonal
$\Rightarrow \quad Q^T Q = I$
Multiply on the right by Q^{-1} :
$$Q^T \cancel{Q Q^{-1}} = I Q^{-1}$$
$$Q^T = Q^{-1}$$

__Ex__: Let Q be orthogonal
Show that Q^{-1} is orthogonal.

To show that M is orthogonal : $M^T M = I$
" Q^{-1} " : $(Q^{-1})^T Q^{-1} = I$

$$(Q^{-1})^T Q^{-1} = (Q^T)^T Q^{-1}$$
$$= Q Q^{-1}$$
$$= I \quad \checkmark$$

__Ex__: Determine values of x, y, z
so that $\begin{bmatrix} \frac{1}{2} & y \\ x & z \end{bmatrix}$ is orthogonal.

(1) $\left\| \begin{bmatrix} \frac{1}{2} \\ x \end{bmatrix} \right\| = 1$

① $\left\| \begin{bmatrix} \frac{1}{2} \\ x \end{bmatrix} \right\| = 1$

$$\sqrt{\frac{1}{4} + x^2} = 1$$

$$\frac{1}{4} + x^2 = 1$$

$$x^2 = \frac{3}{4}$$

$$x = \pm \frac{\sqrt{3}}{2}$$

② If 1^{st} column is $\begin{bmatrix} a \\ b \end{bmatrix}$ then there are 2 options for 2^{nd} column:

$$\begin{bmatrix} -b \\ a \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} b \\ -a \end{bmatrix}$$

Answers:

$x = \frac{\sqrt{3}}{2}$ $\quad \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

$x = -\frac{\sqrt{3}}{2}$ $\quad \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

5.2 Complements and Projections

$W =$ subspace of $\mathbb{R}^n$

e.g. line through $\vec{0}$

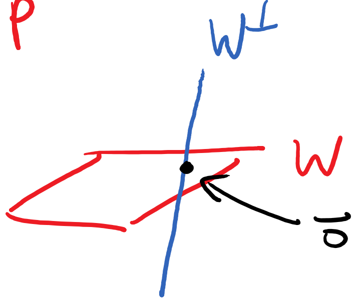
plane "

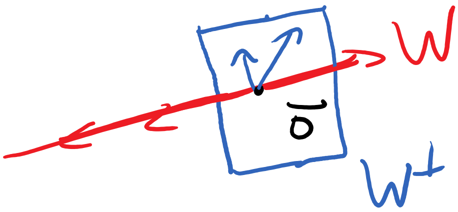
Def

The <u>orthogonal complement of W</u> is

$$W^\perp = \{ \vec{v} \text{ in } \mathbb{R}^n \mid \vec{v}\cdot\vec{w} = 0 \text{ for any } \vec{w} \text{ in } W \}$$

"W perp"





Ex: $W = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right)$

Find a basis for $W^\perp$

$$W^\perp = \left\{ \vec{v} \;\middle|\; \vec{v}\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ and } \vec{v}\cdot\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \;\middle|\; v_1 = 0 \text{ and } v_1 + v_2 + 3v_3 = 0 \right\}$$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \end{array}\right]$$

Shortcut

Basis for $W^\perp$: Put basis vectors of W into <u>rows</u>
Solve the homogeneous system

$$R_2 - R_1 \quad \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

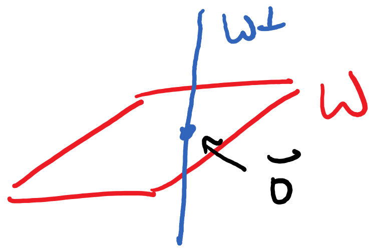
$$\boxed{v_3 = t}$$

$$\boxed{v_1 = 0}$$

$$v_2 + 3v_3 = 0 \rightarrow \boxed{v_2 = -3t}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} t$$

$$\text{Basis of } W^\perp = \left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$



3 Facts about $W^\perp$

1) $\dim W^\perp + \dim W = \dim \mathbb{R}^n$

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[dim = # basis vectors]

Check: $1 + 2 = 3$ in previous example ✓

2) $W \cap W^{\perp} = \{\vec{0}\}$

3) $(W^{\perp})^{\perp} = W$