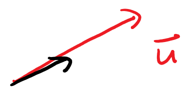
5.1 Orthogonality Cont'd

Normalize a vector: find a unit vector in the same direction

Ex: Normalize $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$



$$\|\vec{u}\| = \sqrt{6}$$

$$\vec{v} = \frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

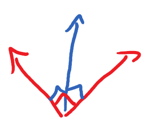
Orthonormal Set: Orthogonal set, with all vectors normalized

Ex: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is orthogonal

$\left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is orthonormal

$$\left\| \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1+1+1} = \sqrt{3}$$

> **FACT**
> Any set of n nonzero orthogonal vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.



Ex: Find an orthonormal basis for $\mathbb{R}^3$ consisting of a vector parallel to $[2, 0, 1]$ and " $[1, 3, -2]$

3 Directions:

$[2, 0, 1]$

$[1, 3, -2]$

$$[2,0,1] \times [1,3,-2] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \vec{i}(-3) - \vec{j}(-5) + \vec{k}(6)$$

$$= [-3, 5, 6]$$

Normalize:

$$\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{70}}\begin{bmatrix} -3 \\ 5 \\ 6 \end{bmatrix} \right\}$$

is orthonormal

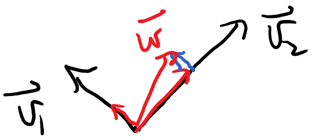
(It's a basis because 3 orthogonal/orthonormal vectors form a basis for $\mathbb{R}^3$.)

Theorem

Given an orthogonal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$,

$$\vec{w} = \text{proj}_{\vec{v}_1} \vec{w} + \text{proj}_{\vec{v}_2} \vec{w} + \ldots + \text{proj}_{\vec{v}_n} \vec{w}$$

(True for any $\vec{w}$)



when basis is orthogonal

$\vec{w} \neq$ sum of projections

Ex: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$

Write $\vec{w} = \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix}$ as a linear combination of the basis vectors.

Long way:
$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} \begin{matrix}1\\1\\4\end{matrix} & \bigcirc & \bigcirc & \begin{matrix}5\\0\\9\end{matrix} \end{array}\right]$$

Short way: $\mathcal{B}$ is an <u>orthogonal</u> basis for $\mathbb{R}^3$

$$\vec{w} = \text{proj}_{\vec{v}_1} \vec{w} + \text{proj}_{\vec{v}_2} \vec{w} + \text{proj}_{\vec{v}_3} \vec{w}$$

$$= \frac{\vec{w} \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 + \dots$$

$$= \frac{41}{18} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Coordinate vector of $\vec{w}$ relative to basis $\mathcal{B}$ is

$$[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 41/18 \\ 5/2 \\ 1/9 \end{bmatrix}$$

(Section 3.5)

Def
An orthogonal matrix is an $n \times n$ matrix whose columns form an orthonormal set

CAUTION

Ex:

$$Q = \begin{array}{c} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{matrix} \end{array}$$

$\vec{v}_1 \cdot \vec{v}_2 = 0, \quad \vec{v}_1 \cdot \vec{v}_3 = 0, \quad \vec{v}_2 \cdot \vec{v}_3 = 0$

$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = 1$

Q is orthogonal

FACT
A square matrix Q is orthogonal if and only if $Q^T Q = I$

$$Q^TQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & & 0 \\ \frac{-1}{\sqrt{2}} & & 0 \\ 0 & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

all vectors are orthogonal

all vectors have length 1

$$= I$$