Test Fri Nov 8

3.4 - 3.6, 4.1 - 4.2 (6 Questions)

Practice Problems on Website

4.3 Eigenvalues Cont'd

5 Facts

(2) If A is invertible and $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

$$A\vec{x} = \lambda\vec{x}$$

$$\Rightarrow \quad A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$$

$$\Rightarrow \quad \vec{x} = A^{-1}\lambda\vec{x}$$

$$\Rightarrow \quad \vec{x} = \lambda A^{-1}\vec{x}$$

$$\Rightarrow \quad \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$$

$$\boxed{A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}}$$

eigenvalue

(3) If $A\vec{x} = \lambda\vec{x}$ then $A^n\vec{x} = \lambda^n\vec{x}$ $(n = 1, 2, 3, \ldots)$

$$A^n\vec{x} = A^{n-1}A\vec{x}$$

$$= A^{n-1}(\lambda \vec{x})$$
$$= \lambda A^{n-1} \vec{x}$$
$$= \lambda^2 A^{n-2} \vec{x}$$
$$\vdots$$
$$= \lambda^n \vec{x}$$

4. If $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of $A+kI$ with eigenvalue $\lambda + k$

$$(A+kI)\vec{x} = A\vec{x} + kI\vec{x}$$
$$= \lambda\vec{x} + k\vec{x}$$
$$= (\lambda + k)\vec{x}$$

5. Suppose A has eigenvectors $\vec{v}_1, \dots, \vec{v}_m$ corresponding to eigenvalues $\lambda_1, \dots, \lambda_m$

$$A^k(c_1\vec{v}_1 + \dots + c_m\vec{v}_m) = c_1\lambda_1^k\vec{v}_1 + \dots + c_m\lambda_m^k\vec{v}_m$$

→ Generalization of Fact 3
Coefficients stay the same

<u>Ex</u>: A is a 2×2 matrix with
$\lambda_1 = -2$ corresponding to eigenvector $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
and $\lambda_2 = 3$ " $\vec{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Find $A^3 \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

1) $\begin{bmatrix} 11 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & -1 & 2 \end{array}\right]$$

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right] \quad \text{RREF}$$

$c_1 = 3$

$c_2 = 4$

2) $A^3 \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

$= A^3 \left(3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$

$= c_1 \lambda_1^3 \vec{x}_1 + c_2 \lambda_2^3 \vec{x}_2$ (Fact 5)

$= 3(-2)^3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4(3)^3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= -24 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 108 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 192 \\ -156 \end{bmatrix}$

4.4 Diagonalization

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Def

An $n \times n$ matrix A is <u>diagonalizable</u> if there exists an invertible matrix P so that $P^{-1}AP = D$ ← diagonal

(easy to compute A^n)

<u>Ex</u>: Find P that diagonalizes

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

→ Find a basis for each eigenspace

$\lambda = 2, 3$ (A is upper triangular)

E_2 : $[A - \lambda I \mid \vec{0}]$

$[A - 2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{ RREF}$$

$x_1 = t$

$x_2 = 0$

$x_3 = 0$

$x_2 = 0$
$x_3 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t \qquad \text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

E_3 :

$$[\, A - 3I \mid \vec{0} \,] \qquad A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 / (-1)$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]} \end{array}$$

$x_2 = s$
$x_3 = t$

$$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3 = -2t$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

P: put eigenvectors into columns

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

D: diagonal matrix
eigenvalues of A on the diagonal,
in same order as P

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Check:

$$P^{-1} A P = D$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} ✓$$