

4.3 Finding Eigenvalues

p1

Ex: Find all eigenvalues of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -4-\lambda & 3 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-4-\lambda)(7-\lambda) = 0$$

$$\lambda = 1, -4, 7$$

FACT

Eigenvalues of an upper/lower triangular or diagonal matrix are the diagonal entries

Ex: Find all eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda) \begin{vmatrix} -5-\lambda & 1 \\ -6 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 7 & 1 \\ 6 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 7 & -5-\lambda \\ 6 & -6 \end{vmatrix} = 0$$

$$(3-\lambda) [(-5-\lambda)(2-\lambda)+6] + [7(2-\lambda)-6] + [-42-6(-5-\lambda)] = 0$$

$$(3-\lambda)(\lambda^2+3\lambda-4) - 7\lambda + 8 + 6\lambda - 12 = 0$$

$$\begin{array}{r}
 3\lambda^2 + 9\lambda - 12 \\
 -\lambda^3 - 3\lambda^2 + 4\lambda \\
 \hline
 -7\lambda + 8 \\
 + 6\lambda - 12 \\
 \hline
 -\lambda^3 + 12\lambda - 16
 \end{array}$$

$$-\lambda^3 + 12\lambda - 16 = 0$$

$$\lambda^3 - 12\lambda + 16 = 0$$

Integer Roots Theorem

If a polynomial has integer coefficients and leading coefficient 1, then any integer roots divide the constant.

Possible roots: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

$$\lambda=1 : \quad 1^3 - 12(1) + 16 \neq 0$$

$$\lambda=-1 : \quad x$$

$$\lambda=2 : \quad 2^3 - 12(2) + 16 = 0$$

$\lambda=2$ is a root

$\Rightarrow \lambda-2$ is a factor of $\lambda^3 - 12\lambda + 16$

$$\begin{array}{r}
 \lambda^2 + 2\lambda - 8 \\
 \lambda - 2 \quad \overline{) \lambda^3 + 0\lambda^2 - 12\lambda + 16} \\
 - (\lambda^3 - 2\lambda^2) \\
 \hline
 2\lambda^2 - 12\lambda + 16 \\
 - (2\lambda^2 - 4\lambda) \\
 \hline
 -8\lambda + 16 \\
 - (-8\lambda + 16) \\
 \hline
 0
 \end{array}$$

$$\lambda^3 - 12\lambda + 16 = 0$$

$$(\lambda-2)(\lambda^2 + 2\lambda - 8) = 0$$

$$(\lambda-2)(\lambda-2)(\lambda+4) = 0$$

$$(\lambda-2)^2(\lambda+4) = 0$$

$$\lambda = 2, 4$$

Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$E_{-4} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

FACT

When bases for different eigenspaces are combined, the new set is linearly independent

Useful in Section 4.4

Ex: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent

DEF

- Characteristic equation of A $(\lambda - 2)^2(\lambda + 4) = 0$
- Algebraic multiplicity of an eigenvalue λ_i is the exponent on $(\lambda - \lambda_i)$ in characteristic equation
- Geometric multiplicity of an eigenvalue is the number of basis vectors in the eigenspace

Ex:	<u>Eigenvalue</u>	<u>Alg. Mult.</u>	<u>Geo. Mult.</u>
	$\lambda = 2$	2	1
	$\lambda = -4$	1	1

Ex: A is 5×5

p5

$$|A - \lambda I| = (7-\lambda)^3 (9-\lambda)^2$$

Basis for $E_7 = \left\{ \begin{bmatrix} \end{bmatrix} \right\}_3$

Basis for $E_9 = \left\{ \begin{bmatrix} \end{bmatrix}, \begin{bmatrix} \end{bmatrix} \right\}_3$

<u>Eigenvalue</u>	<u>Alg. Mult.</u>	<u>Geo. Mult.</u>
$\lambda=7$	3	1
$\lambda=9$	2	2

FACT

For each eigenvalue:
geometric multiplicity \leq algebraic multiplicity

When geo. mult. = alg. mult. for all

eigenvalues, A has a nice property

(Section 4.4)

5 Facts about Eigenvectors

p6

- ① A is invertible if and only if 0 is not an eigenvalue of A

Proof:

$$\begin{aligned} & A \text{ is invertible} \\ \Leftrightarrow & \det A \neq 0 \\ \Leftrightarrow & \det(A - 0I) \neq 0 \\ \Leftrightarrow & 0 \text{ is not an eigenvalue of } A \end{aligned}$$

- ② If A is invertible and $A\bar{x} = \lambda\bar{x}$
then \bar{x} is an eigenvector of A^{-1}
with eigenvalue $\frac{1}{\lambda}$

$$A\bar{x} = \lambda\bar{x}$$

$$\Rightarrow A^{-1}A\bar{x} = A^{-1}\lambda\bar{x}$$

$$\Rightarrow I\bar{x} = A^{-1}\lambda\bar{x}$$

$$\Rightarrow \bar{x} = A^{-1}\lambda\bar{x}$$

$$\Rightarrow \bar{x} = \lambda A^{-1}\bar{x}$$

$$\Rightarrow \frac{1}{\lambda}\bar{x} = A^{-1}\bar{x} \quad (\lambda \neq 0 \text{ by above})$$

$$\Rightarrow A^{-1}\bar{x} = \frac{1}{\lambda}\bar{x}$$

P7

③ If $A\bar{x} = \lambda\bar{x}$ then $A^n\bar{x} = \lambda^n\bar{x}$
for $n=2, 3, \dots$

$$\begin{aligned} A^n\bar{x} &= A^{n-1}A\bar{x} \\ &= A^{n-1}(\lambda\bar{x}) \\ &= \lambda A^{n-1}\bar{x} \\ &\vdots \\ &= \lambda^n\bar{x} \end{aligned}$$

④ If $A\bar{x} = \lambda\bar{x}$ then \bar{x} is an eigenvector
of $A+kI$ with eigenvalue $\lambda+k$

$$\begin{aligned} (A+kI)\bar{x} &= A\bar{x} + kI\bar{x} \\ &= \lambda\bar{x} + k\bar{x} \\ &= (\lambda+k)\bar{x} \end{aligned}$$

⑤ Suppose A has eigenvectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_m$
Corresponding to $\lambda_1, \lambda_2, \dots, \lambda_m$.

$$A^k(c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_m\bar{v}_m) = c_1\lambda_1^k\bar{v}_1 + \dots + c_m\lambda_m^k\bar{v}_m$$

- Generalization of Fact ③
- Coefficients are preserved

Ex: A is a 2×2 matrix with

$$\lambda_1 = -2 \quad \bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 3 \quad \bar{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Find $A^3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

1) Let $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} c_1 & c_2 & \\ 1 & 2 & 1 \\ 2 & -1 & 2 \end{array} \right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array} \right]$$

2) $A^3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = A^3 (3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ -1 \end{bmatrix})$

$$= 3 \lambda_1^3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \lambda_2^3 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= -24 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 108 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 192 \\ -156 \end{bmatrix}$$

③ Not to underestimate
bravery in the field.