1. [4 marks] Find all the eigenvalues of $A = \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 2 \\ 4 & 7-\lambda \end{vmatrix}$$ 1

$$= (5-\lambda)(7-\lambda) - 8$$

$$= \lambda^2 - 12\lambda + 35 - 8$$

$$= \lambda^2 - 12\lambda + 27$$ 1

Set $\lambda^2 - 12\lambda + 27 = 0$

$(\lambda - 3)(\lambda - 9) = 0$

$\boxed{\lambda = 3, 9}$ 2

2. [5 marks] Find a basis for the eigenspace corresponding to $\lambda = 3$ for

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 1 & 5 & 3 & 4 \\ 3 & 3 & 6 & 3 \\ 1 & 3 & 5 & 10 \end{bmatrix}.$$

Solve $[A - \lambda I \mid \vec{0}]$

$[A - 3I \mid \vec{0}]$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 3 & 3 & 3 & 3 & 0 \\ 1 & 3 & 5 & 7 & 0 \end{array}\right]$$

1

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ \\ \\ R_4 - 2R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2

$\boxed{x_3 = s}$ $\boxed{x_4 = t}$

$x_1 - x_3 - 2x_4 = 0$

$\boxed{x_1 = s + 2t}$

$x_2 + 2x_3 + 3x_4 = 0$

$\boxed{x_2 = -2s - 3t}$

2

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3. [3 marks] Use Cramer's Rule to find y.

$$\begin{aligned} 19x - 28y &= 370 \\ -17x + 44y &= -440 \end{aligned}$$

$$y = \frac{|A_2|}{|A|}$$

1

$$= \frac{\begin{vmatrix} 19 & 370 \\ -17 & -440 \end{vmatrix}}{\begin{vmatrix} 19 & -28 \\ -17 & 44 \end{vmatrix}}$$

$$= \frac{-2070}{360}$$

2

$$= \frac{-23}{4} \text{ or } -5.75$$

4. [6 marks] Let $W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ \text{[redacted]} \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -3 \\ \text{[redacted]} \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \\ \text{[redacted]} \end{bmatrix} \right)$.

Use the Gram-Schmidt procedure to find an orthogonal basis for W.

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ Partial basis $X = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

1

$\vec{v}_2 = \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix} - \cancel{\frac{6}{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\frac{3}{7}$

$7\vec{v}_2 = 7\begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$7\vec{v}_2 = \begin{bmatrix} 4 \\ 43 \\ -30 \end{bmatrix}$ 1

Partial basis $X = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 43 \\ -30 \end{bmatrix} \right\}$

$\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} - \cancel{\frac{2}{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{197}{2765} \begin{bmatrix} 4 \\ 43 \\ -30 \end{bmatrix}$ 2

$\frac{1}{7}$

$2765\,\vec{v}_3 = 2765 \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} - 395 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 197 \begin{bmatrix} 4 \\ 43 \\ -30 \end{bmatrix}$

$= \begin{bmatrix} 4347 \\ -966 \\ -805 \end{bmatrix}$ 1

1

Basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 43 \\ -30 \end{bmatrix}, \begin{bmatrix} 4347 \\ -966 \\ -805 \end{bmatrix} \right\}$

$$\begin{array}{ccc} \vec{w_1} & \vec{w_2} & \vec{w_3} \end{array}$$

5. [3 marks] $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \}$ is an orthogonal basis for $\mathbb{R}^3$.

Write $\mathbf{v} = \begin{bmatrix} 6 \\ -7 \\ 8 \end{bmatrix}$ as a linear combination of the three basis vectors.

Orthogonal basis $\Rightarrow$

$$\vec{v} = \frac{\vec{v} \cdot \vec{w_1}}{\|\vec{w_1}\|^2} \vec{w_1} + \frac{\vec{v} \cdot \vec{w_2}}{\|\vec{w_2}\|^2} \vec{w_2} + \frac{\vec{v} \cdot \vec{w_3}}{\|\vec{w_3}\|^2} \vec{w_3}$$

$$= \frac{7}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{17}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \frac{13}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

3

6. [4 marks] Find a 2×2 matrix A with eigenvalue $\lambda_1 = -6$ corresponding to eigenvector $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and eigenvalue $\lambda_2 = 11$ corresponding to eigenvector $\mathbf{v}_2 = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$. Simplify your answer.

$$A = \lambda_1 \bar{q}_1 \bar{q}_1^T + \lambda_2 \bar{q}_2 \bar{q}_2^T$$

where $\{\bar{q}_1, \bar{q}_2\}$ is an orthonormal set of eigenvectors

$$\vec{q}_1 = \frac{1}{\sqrt{58}} \begin{bmatrix} 3 \\ 7 \end{bmatrix} \qquad \vec{q}_2 = \frac{1}{\sqrt{58}} \begin{bmatrix} -7 \\ 3 \end{bmatrix}$$ 1

orthonormal ✓

$$A = \frac{-6}{58} \begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} 3 & 7 \end{bmatrix} + \frac{11}{58} \begin{bmatrix} -7 \\ 3 \end{bmatrix} \begin{bmatrix} -7 & 3 \end{bmatrix}$$ 2

$$= \frac{-6}{58} \begin{bmatrix} 9 & 21 \\ 21 & 49 \end{bmatrix} + \frac{11}{58} \begin{bmatrix} 49 & -21 \\ -21 & 9 \end{bmatrix}$$

$$= \frac{1}{58} \begin{bmatrix} 485 & -357 \\ -357 & -195 \end{bmatrix}$$ 1