1. [6 marks] Find A^{-1} and use it to solve the system.

$$\begin{aligned} x + \ \ y - 6z &= 5 \\ 3x + \ 4y + 7z &= -4 \\ 4x + 4y - 22z &= 18 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -6 & 1 & 0 & 0 \\ 3 & 4 & 7 & 0 & 1 & 0 \\ 4 & 4 & -22 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-3R_1 \\ R_3-4R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & -6 & 1 & 0 & 0 \\ 0 & 1 & 25 & -3 & 1 & 0 \\ 0 & 0 & 2 & -4 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1-R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -31 & 4 & -1 & 0 \\ 0 & 1 & 25 & -3 & 1 & 0 \\ 0 & 0 & 2 & -4 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3/2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -31 & 4 & -1 & 0 \\ 0 & 1 & 25 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & \frac{1}{2} \end{array}\right]$$

$$\begin{array}{l} R_1+31R_3 \\ R_2-25R_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -58 & -1 & \frac{31}{2} \\ 0 & 1 & 0 & 47 & 1 & -\frac{25}{2} \\ 0 & 0 & 1 & -2 & 0 & \frac{1}{2} \end{array}\right]$$

A^{-1}

4

$A\bar{x} = \bar{b}$

1

$\bar{x} = A^{-1}\bar{b}$

$$\bar{x} = \begin{bmatrix} -58 & -1 & \frac{31}{2} \\ 47 & 1 & -\frac{25}{2} \\ -2 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 18 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ -1 \end{bmatrix}$$

1

2. [4 marks] Solve the following system using the LU method.

$$\begin{bmatrix} 1 & 0 \\ -19 & 1 \end{bmatrix} \begin{bmatrix} 29 & -37 \\ 0 & 87 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -235 \\ 4813 \end{bmatrix}.$$

1) $\underbrace{LU\vec{x}}_{\vec{y}} = \vec{b}$

Solve $L\vec{y} = \vec{b}$

$$\begin{array}{cc} y_1 & y_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & -235 \\ -19 & 1 & 4813 \end{array}\right]$$

$\boxed{y_1 = -235}$

1

$-19y_1 + y_2 = 4813$

$4465 + y_2 = 4813$

1

$\boxed{y_2 = 348}$

2) $U\vec{x} = \vec{y}$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 29 & -37 & -235 \\ 0 & 87 & 348 \end{array}\right]$$

$87x_2 = 348$

1

$\boxed{x_2 = 4}$

$29x_1 - 37x_2 = -235$

$29x_1 - 148 = -235$

$29x_1 = -87$

1

$\boxed{x_1 = -3}$

$\boxed{\vec{x} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}}$

3. [4 marks] $A = \begin{bmatrix} 8 & 6 & 4 & 2 \\ 4 & 3 & 2 & 1 \\ 0 & 5 & 6 & 7 \\ 0 & 4 & 5 & 6 \end{bmatrix}$ has RREF$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for:

a) the row space of A

$\{[1\ 0\ 0\ 0], [0\ 1\ 0\ {-1}], [0\ 0\ 1\ 2]\}$

1

b) the column space of A

$\left\{ \begin{bmatrix} 8 \\ 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 6 \\ 5 \end{bmatrix} \right\}$

1

c) the null space of A

Solve $A\bar{x} = \bar{0}$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

1

$\boxed{x_4 = t}$

$x_3 + 2x_4 = 0$

$\boxed{x_3 = -2t}$

$x_2 - x_4 = 0$

$\boxed{x_2 = t}$

$\boxed{x_1 = 0}$

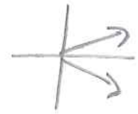
$$\bar{x} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} t$$

1

Basis for null(A) = $\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

4. [6 marks] Find the standard matrix for the following transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$. T first reflects a vector in the x-axis, then rotates it by $60°$ counter-clockwise, then reflects it in the line $y = x$. Simplify your answer.

$$[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ (1)



$$[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 60°$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$ (2)

$$[T_3] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ (1)

$$[T] = [T_3][T_2][T_1]$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$ (1)

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$ (1)

−0.5 for sign errors

5. [5 marks] Let $A = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$.

a) Find A^3

$$A^2 = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+ab & 2a \\ 2b & ab+1 \end{bmatrix}$$ 1

$$A^3 = A^2 A$$

$$= \begin{bmatrix} 1+ab & 2a \\ 2b & ab+1 \end{bmatrix}\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$ 2

$$= \begin{bmatrix} 1+ab+2ab & a+a^2b+2a \\ 2b+ab^2+b & 2ab+ab+1 \end{bmatrix} = \begin{bmatrix} 1+3ab & 3a+a^2b \\ 3b+ab^2 & 1+3ab \end{bmatrix}$$

b) Find a condition on a and b so that $A^2 = 2A$.

$$A^2 = 2A$$

$$\begin{bmatrix} 1+ab & 2a \\ 2b & ab+1 \end{bmatrix} = \begin{bmatrix} 2 & 2a \\ 2b & 2 \end{bmatrix}$$ 1

$$\boxed{\begin{array}{c} 1+ab=2 \\ \text{or} \\ ab=1 \end{array}}$$ 1

I will accept specific instances of this,
e.g. $a=b=1$ or $a=b=-1$
etc.