25

$$D = P^{-1} A P$$

$$P D P^{-1} = A$$

$$P D^{7} P^{-1} = A^{7}$$

$$D^{7} = \begin{bmatrix} 2^{7} & 0 \\ 0 & (-3)^{7} \end{bmatrix} = \begin{bmatrix} 128 & 0 \\ 0 & -2187 \end{bmatrix}$$

$$P^{-1} = -\frac{1}{8}\begin{bmatrix} 1 & -1 \\ -3 & -5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix}$$

$$\begin{aligned} A^{7} &= P D^{7} P^{-1} \\ &= \frac{1}{8}\begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 128 & 0 \\ 0 & -2187 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -128 & 128 \\ -6561 & -10935 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} -5921 & -11575 \\ -6945 & -10551 \end{bmatrix} \end{aligned}$$

(26) Call the basis vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$

Since the basis is orthogonal:

$$\vec{v} = \frac{\vec{v}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} + \frac{\vec{v}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} + \frac{\vec{v}\cdot\vec{v_3}}{\vec{v_3}\cdot\vec{v_3}}\vec{v_3}$$

$$= \frac{9}{2}\vec{v_1} + \frac{2}{3}\vec{v_2} - \frac{11}{6}\vec{v_3}$$

$$[\vec{v}]_\beta = \left[\frac{9}{2}, \frac{2}{3}, \frac{-11}{6}\right]$$

(27) The basis for W is orthogonal

Call the basis vectors $\vec{w_1}$ and $\vec{w_2}$

$$\text{proj}_W \vec{u} = \text{proj}_{\vec{w_1}} \vec{u} + \text{proj}_{\vec{w_2}} \vec{u}$$

$$= \frac{\vec{u}\cdot\vec{w_1}}{\vec{w_1}\cdot\vec{w_1}}\vec{w_1} + \frac{\vec{u}\cdot\vec{w_2}}{\vec{w_2}\cdot\vec{w_2}}\vec{w_2}$$

$$= \frac{41}{29}\begin{bmatrix}3\\2\\4\end{bmatrix} - \frac{7}{9}\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

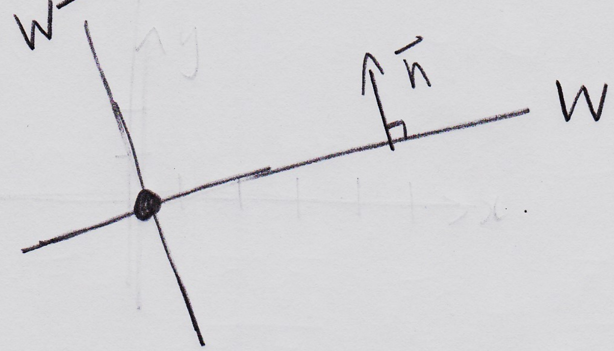
$$= \frac{369}{261}\begin{bmatrix}3\\2\\4\end{bmatrix} - \frac{203}{261}\begin{bmatrix}2\\1\\-2\end{bmatrix}$$

$$= \frac{1}{261}\begin{bmatrix}701\\535\\1882\end{bmatrix}$$

and

$$\text{perp}_W \vec{u} = \vec{u} - \text{proj}_W \vec{u} = \frac{261}{261}\begin{bmatrix}1\\5\\7\end{bmatrix} - \frac{1}{261}\begin{bmatrix}701\\535\\1882\end{bmatrix} = \frac{1}{261}\begin{bmatrix}-440\\770\\-55\end{bmatrix}$$

(28)



Basis for $W^{\perp} = \{ \vec{n} \}$

$= \left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$

(29) Solve $A\bar{x} = \bar{0}$, where A has the basis vectors in its <u>rows</u>

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 10 & 7 \end{bmatrix} \bar{x} = \bar{0}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 2 & 3 & 10 & 7 & 0 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 6 & 1 & 0 \end{array}\right]$$

$$R_1 - R_2 \quad \left[\begin{array}{cccc|c} 1 & 0 & -4 & 2 & 0 \\ 0 & 1 & 6 & 1 & 0 \end{array}\right]$$

$x_3 = s \qquad x_4 = t$

$x_1 - 4x_3 + 2x_4 = 0 \rightarrow x_1 = 4s - 2t$

$x_2 + 6x_3 + x_4 = 0 \rightarrow x_2 = -6s - t$

$$\bar{x} = \begin{bmatrix} 4 \\ -6 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $W^{\perp} = \left\{ \begin{bmatrix} 4 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

30 Gram-Schmidt

$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ partial basis $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$4\bar{v}_2 = \begin{bmatrix} 0 \\ 4 \\ 4 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ partial basis $X = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\bar{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{(-2)}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$6\bar{v}_3 = 6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -2 \\ -2 \end{bmatrix}$ Basis $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -2 \\ -2 \end{bmatrix} \right\}$

(31) $A = \lambda_1 \bar{q}_1 \bar{q}_1^T + \lambda_2 \bar{q}_2 \bar{q}_2^T$

where $\bar{q}_1, \bar{q}_2$ are <u>orthonormal</u> eigenvectors

$\lambda_1 = 2 \quad \bar{q}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = -2 \quad \bar{q}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$A = 2 \cdot \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} - 2 \cdot \frac{1}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix}$$

(32) $x^* = (A^TA)^{-1}A^T\vec{b}$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 10 & 150 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{500}\begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix}$$

$$x^* = (A^TA)^{-1}A^T\vec{b}$$

$$= \frac{1}{500}\begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$$= \frac{1}{500}\begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 55 \end{bmatrix}$$

$$= \frac{1}{500}\begin{bmatrix} 350 \\ 160 \end{bmatrix}$$

or $\begin{bmatrix} \frac{7}{10} \\ \frac{8}{25} \end{bmatrix}$

(33) Best-fit line $y = a_0 + a_1 x$

$$1(a_0) + x(a_1) = y$$

$$\begin{array}{cc} 1 & x \end{array} \qquad \qquad \qquad y$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

$$x^* = (A^T A)^{-1} A^T \bar{b}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix}$$

$$x^* = (A^T A)^{-1} A^T \bar{b}$$

$$= \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 30 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 17 \\ 51 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 0 \\ 34 \end{bmatrix}$$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{17}{10} \end{bmatrix} \rightarrow y = a_0 + a_1 x \rightarrow \boxed{y = \frac{17}{10} x}$$

(34) $$z = \frac{(3-7i)}{(c+di)} \cdot \frac{(c-di)}{(c-di)}$$

$$= \frac{3c - 7d + i(-7c - 3d)}{c^2 + d^2}$$

$$= \frac{3c - 7d}{c^2+d^2} - \frac{(7c+3d)}{c^2+d^2} i$$

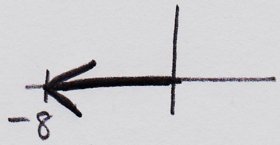
(35) $$z = |z|[\cos\theta + i\sin\theta]$$

$$|z| = \sqrt{5^2+12^2} = 13$$

$$\theta = \tan^{-1}\frac{12}{5}$$

$$z = 13\left[\cos\left(\tan^{-1}\frac{12}{5}\right) + i\sin\left(\tan^{-1}\frac{12}{5}\right)\right]$$

(36) $z = -8$



in polar form $z = 8\left[\cos\pi + i\sin\pi\right]$

$$z^{1/3} = 2\left[\cos\frac{\pi + 2\pi\alpha}{3} + i\sin\frac{\pi+2\pi\alpha}{3}\right]$$

$\alpha = 0, 1, 2$

α	$\frac{\pi + 2\pi\alpha}{3}$
0	$\pi/3$
1	$\frac{3\pi}{3} = \pi$
2	$\frac{5\pi}{3}$

$$z_1^{1/3} = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = 1 + \sqrt{3}\,i$$

$$z_2^{1/3} = 2\left[\cos\pi + i\sin\pi\right] = -2$$

$$z_3^{1/3} = 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] = 1 - \sqrt{3}\,i$$