(13) $BB^T = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -3 \\ -1 & 4 \end{bmatrix}$

$= \begin{bmatrix} 5 & 2 \\ 2 & 34 \end{bmatrix}$

$(BB^T)^{-1} = \frac{1}{166} \begin{bmatrix} 34 & -2 \\ -2 & 5 \end{bmatrix}$

(14) Elementary matrices act on the <u>left</u> of A :

$E_3 E_2 E_1 A = I$

$A = E_1^{-1} E_2^{-1} E_3^{-1}$

$$A = \underset{R_2 \leftrightarrow R_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{2R_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{R_1 \to R_1 + 3R_3}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

(15) PART 1: Find $A = LU$

$A \to REF$ using $R_i - kR_j$. Record k.

$$\begin{bmatrix} 3 & 5 & 0 \\ 6 & -4 & 2 \\ 3 & -2 & 0 \end{bmatrix}$$

$$\begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \begin{bmatrix} 3 & 5 & 0 \\ 0 & -14 & 2 \\ 0 & -7 & 0 \end{bmatrix} \begin{array}{l} \\ k=2 \\ k=1 \end{array}$$

$$\begin{array}{l} \\ \\ R_3 - \frac{1}{2}R_2 \end{array} \begin{bmatrix} 3 & 5 & 0 \\ 0 & -14 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{array}{l} \\ \\ k = \frac{1}{2} \end{array}$$

REF ✓

$$A = LU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 0 \\ 0 & -14 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

PART 2: Solve using $A = LU$

$$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$

1) Solve $L\vec{y} = \vec{b}$

$$\begin{array}{ccc|c} y_1 & y_2 & y_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 23 \\ 2 & 1 & 0 & -8 \\ 1 & \frac{1}{2} & 1 & -5 \end{array}\right]$$

$y_1 = 23$

$2y_1 + y_2 = -8 \to 46 + y_2 = -8 \to y_2 = -54$

$y_1 + \frac{1}{2}y_2 + y_3 = -5 \to 23 - 27 + y_3 = -5$

$\to y_3 = -1$

2) Solve $U\bar{x} = \bar{y}$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & 5 & 0 & 23 \\ 0 & -14 & 2 & -54 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$-x_3 = -1 \rightarrow x_3 = 1$$

$$-14x_2 + 2x_3 = -54 \rightarrow -14x_2 + 2 = -54 \rightarrow x_2 = 4$$

$$3x_1 + 5x_2 = 23 \rightarrow 3x_1 + 20 = 23 \rightarrow x_1 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

16) a) nonzero rows of RREF of A

$\{[1\ 0\ -1\ -2], [0\ 1\ 2\ 3]\}$

b) $\text{row}(A) = \text{col}(A^T)$

Use columns 1 and 3 of A^T

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

c) Use columns 1 and 2 of A

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 4 \end{bmatrix} \right\}$

d) $\text{null}(A) = \{ \bar{x} \text{ such that } A\bar{x} = \bar{0} \}$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$

$x_4 = t$

$x_1 - x_3 - 2x_4 = 0 \rightarrow x_1 = s + 2t$

$x_2 + 2x_3 + 3x_4 = 0 \rightarrow x_2 = -2s - 3t$

$\bar{x} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ Basis $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

(17) rank(A) could be 0, 1, 2, 3, 4

rank + nullity = # columns

nullity = 6 − rank

Possible values for nullity: 2, 3, 4, 5, 6

(18) $[S] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{array}{cc} \uparrow & \uparrow \\ \text{Image of} & \text{Image of} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array}$

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta = \frac{\pi}{3}}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right) = [S][T]\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1-\sqrt{3} \\ \sqrt{3}+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-\sqrt{3} \\ -\sqrt{3}-1 \end{bmatrix}$$

(19) Find c_1 and c_2 so that

$$c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}6\\8\end{bmatrix}$$

$$\begin{array}{ccc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 0 & 6 \\ 1 & 1 & 8 \end{array}\right]$$

$$R_1/2 \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & 1 & 8 \end{array}\right]$$

$$R_2-R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$$

$$c_1 = 3 \quad c_2 = 5$$

Therefore $\begin{bmatrix}6\\8\end{bmatrix} = 3\begin{bmatrix}2\\1\end{bmatrix} + 5\begin{bmatrix}0\\1\end{bmatrix}$

$$T\left(\begin{bmatrix}6\\8\end{bmatrix}\right) = T\left(3\begin{bmatrix}2\\1\end{bmatrix} + 5\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= T\left(3\begin{bmatrix}2\\1\end{bmatrix}\right) + T\left(5\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 3T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) + 5T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

T is linear

$$= 3\begin{bmatrix}5\\6\end{bmatrix} + 5\begin{bmatrix}3\\4\end{bmatrix}$$

$$= \begin{bmatrix}30\\38\end{bmatrix}$$

(20) $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$

$T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$

Therefore $T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) \neq T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

(21) Cramer's Rule

$$y = \frac{|A_y|}{|A|}$$

$$|A_y| = \begin{vmatrix} 3 & 32 & 4 \\ 5 & 39 & -1 \\ 6 & 38 & 1 \end{vmatrix} = 3(77) - 32(11) + 4(-44) = -297$$

$$|A| = \begin{vmatrix} 3 & -1 & 4 \\ 5 & -2 & -1 \\ 6 & 2 & 1 \end{vmatrix} = 3(0) + 1(11) + 4(22) = 99$$

$$y = \frac{-297}{99} = -3$$

(22) Yes if and only if $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 5 & 3 & -1 \end{vmatrix} \neq 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 5 & 3 & -1 \end{vmatrix} = 1(-12) - 1(-21) + 1(3)$$

$$= 12$$

$$\neq 0$$

$\boxed{\text{Yes}}$

(23) a) Set $\det(A - \lambda I) = 0$

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & -2 & -2 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(4-\lambda)(1-\lambda) + 2\right]$$

$$= (2-\lambda)\left[\lambda^2 - 5\lambda + 6\right]$$

$$= (2-\lambda)(\lambda-2)(\lambda-3)$$

$$= -(\lambda-2)^2(\lambda-3)$$

Set $|A - \lambda I| = 0$: $\lambda = 2, 3$

b)

λ	alg. multiplicity
2	2
3	1

Find a basis for E_2:

Solve $[A-2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$R_1/2 \quad \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

$$R_3-R_1 \quad \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = s \quad x_3 = t$

$x_1 - x_2 - x_3 = 0 \rightarrow x_1 = s+t$

$$\vec{x} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(24) If $\begin{bmatrix}3\\7\end{bmatrix} = c_1\vec{v_1} + c_2\vec{v_2}$ ($\vec{v_1}, \vec{v_2}$: eigenvectors)

then $A^{-5}\begin{bmatrix}3\\7\end{bmatrix} = c_1\lambda_1^{-5}\vec{v_1} + c_2\lambda_2^{-5}\vec{v_2}$

Find c_1 and c_2: $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}3\\7\end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 7 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & 4 \end{array}\right]$$

$$R_2/(-2) \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

$$R_1 - R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

$c_1 = 5 \quad c_2 = -2$

$$\begin{aligned} A^{-5}\begin{bmatrix}3\\7\end{bmatrix} &= c_1\lambda_1^{-5}\vec{v_1} + c_2\lambda_2^{-5}\vec{v_2} \\ &= 5\left(\tfrac{1}{2}\right)^{-5}\begin{bmatrix}1\\1\end{bmatrix} - 2(-1)^{-5}\begin{bmatrix}1\\-1\end{bmatrix} \\ &= 160\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}1\\-1\end{bmatrix} \\ &= \begin{bmatrix}162\\158\end{bmatrix} \end{aligned}$$