① $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

$$-3 = \sqrt{2}\,\sqrt{29}\,\cos\theta$$

$$\cos\theta = \frac{-3}{\sqrt{2}\,\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{2}\,\sqrt{29}}\right)$$

$$\approx 113.2^\circ$$

② $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) + (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$

$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$+ \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= 2(\vec{u}\cdot\vec{u}) + 2(\vec{v}\cdot\vec{v})$$

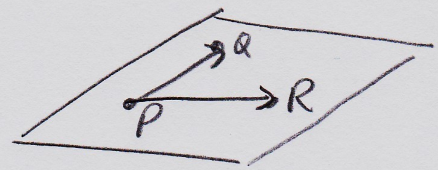
$$= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

(3) Start with 3 points on the plane

$P = (2,0,0) \qquad Q = (0,2,0) \qquad R = (0,0,2)$

Find 2 (non-parallel) direction vectors

$$\vec{u} = \overrightarrow{PQ} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$



$$\vec{v} = \overrightarrow{PR} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

Vector form : $\bar{x} = \bar{p} + s\bar{u} + t\bar{v}$

$$\bar{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

(4) $2x + 3y - z = 0$ has $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

Desired plane also has $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

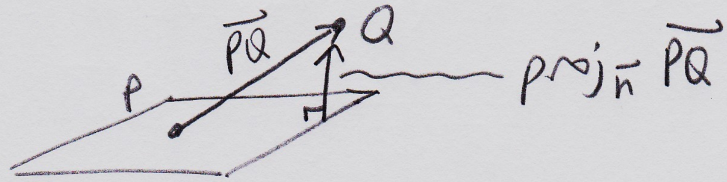
Normal form : $\vec{n} \cdot \bar{x} = \vec{n} \cdot \bar{p}$

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

General form : $2x + 3y - z = 7$

(5) Choose any point P on the plane

distance $= \| \mathrm{proj}_{\vec{n}} \overrightarrow{PQ} \|$



Let $P = (0,0,0)$

$$\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

$$\vec{n} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

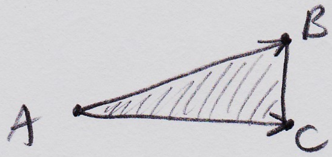
$$\mathrm{proj}_{\vec{n}} \overrightarrow{PQ} = \frac{\vec{n} \cdot \overrightarrow{PQ}}{\|\vec{n}\|^2} \vec{n}$$

$$= \frac{13}{35} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$\text{distance} = \left\| \frac{13}{35} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\|$$

$$= \frac{13}{35} \sqrt{35}$$

6



$$\text{area} = \frac{1}{2} \| \vec{AB} \times \vec{AC} \|$$

$$\vec{AB} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 2 \\ 4 \\ 9 \end{bmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 5 \\ 2 & 4 & 9 \end{vmatrix}$$

$$= 7\vec{i} + \vec{j} - 2\vec{k}$$

$$\| \vec{AB} \times \vec{AC} \| = \sqrt{54}$$

$$= 3\sqrt{6}$$

$$\text{area} = \frac{3\sqrt{6}}{2}$$

(7)

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & a \\ 2 & 1 & 3 & b \\ 7 & -19 & 13 & c \end{array}\right]$$

Each zero row of the REF leads to a condition on a, b, c

$$\begin{array}{l} \\ R_2-2R_1 \\ R_3-7R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 1 & a \\ 0 & -9 & 1 & b-2a \\ 0 & -54 & 6 & c-7a \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3-6R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 1 & a \\ 0 & -9 & 1 & b-2a \\ 0 & 0 & 0 & 5a-6b+c \end{array}\right]$$

REF ✓

$$\boxed{\begin{aligned} & c-7a-6(b-2a) \\ &= c-7a-6b+12a \\ &= 5a-6b+c \end{aligned}}$$

$5a-6b+c=0$ guarantees that the system is consistent

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$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - kR_1 \end{array} \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right]$$

$1-k^2 \neq 0$

$1-k^2 = 0$

$(1-k)(1+k) = 0$

$$\begin{array}{c} \\ \frac{R_2}{1-k^2} \end{array} \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{1-k}{1-k^2} \end{array}\right]$$

unique solution

$k=1$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

infinitely-many solutions

$k=-1$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

no solution

No solution if $k=-1$

1 solution if $k \neq \pm 1$

Infinitely-many solutions if $k=1$

(9) Set $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$$s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -7 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} s & t & \# \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 4 \\ -1 & -1 & -4 \\ 2 & -1 & -7 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_2+R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & -3 & -15 \end{array}\right]$$

$$\begin{array}{l} R_3/(-3) \\ \text{and } R_2 \leftrightarrow R_3 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

$$R_1-R_2 \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

Sub $s=-1$ <u>or</u> $t=5$ into original equation

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ The point is $(0,3,1)$.

10 a) Notice $(\vec{u}+\vec{v})+(\vec{u}-\vec{v}) = 2\vec{u}$

Therefore $\vec{u} = \frac{1}{2}(\vec{u}+\vec{v}) + \frac{1}{2}(\vec{u}-\vec{v}) + 0(\vec{u}+\vec{v}+\vec{w})$

$\vec{u}$ is in the span ✓

b) Notice $(\vec{u}+\vec{v})-(\vec{u}-\vec{v}) = 2\vec{v}$

therefore $\vec{v} = \frac{1}{2}(\vec{u}+\vec{v}) - \frac{1}{2}(\vec{u}-\vec{v}) + 0(\vec{u}+\vec{v}+\vec{w})$

$\vec{v}$ is in the span ✓

c) Notice $\vec{w} = (\vec{u}+\vec{v}+\vec{w}) - (\vec{u}+\vec{v})$

Therefore $\vec{w} = -1(\vec{u}+\vec{v}) + 0(\vec{u}-\vec{v}) + 1(\vec{u}+\vec{v}+\vec{w})$

$\vec{w}$ is in the span ✓

(11) Any span contains the point (0,0,0) (because $\vec{0}$ is in every span).

We'll use $\vec{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= -\vec{i} + 2\vec{j} - \vec{k}$$

$$= \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

Normal form: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$$\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

General form: $-x + 2y - z = 0$

or $x - 2y + z = 0$

(12) Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be in the span.

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Each zero row of the REF leads to a condition on a, b, c, d

$$\begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 1 & 0 & b \\ 1 & 0 & 0 & c \\ 1 & 0 & 1 & d \end{array}\right]$$

$R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 0 & 0 & b-a \\ 0 & -1 & 0 & c-a \\ 0 & -1 & 1 & d-a \end{array}\right]$$

$R_4 / (-1)$ and $R_2 \leftrightarrow R_4$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & a-d \\ 0 & -1 & 0 & c-a \\ 0 & 0 & 0 & b-a \end{array}\right]$$

$R_1 - R_2$, $R_3 + R_2$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & d \\ 0 & 1 & -1 & a-d \\ 0 & 0 & -1 & c-d \\ 0 & 0 & 0 & b-a \end{array}\right]$$

REF ✓

Restriction: $b - a = 0$

$b = a$

Therefore the span is $\left\{ \begin{bmatrix} a & a \\ c & d \end{bmatrix} \right\}$