Math 251 Review Problems

1. Find the angle between $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

2. Let $\mathbf{u}$ and $\mathbf{v}$ be in $\mathbb{R}^n$.
Show that $||\mathbf{u}+\mathbf{v}||^2 + ||\mathbf{u}-\mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}|^2$

3. Find the vector form of the plane $x + y + z = 2$

4. Consider the plane through $(3, 2, 5)$ that is parallel to the plane $2x + 3y - z = 0$. Find the general form of the plane.

5. Find the distance between $Q = (1, 4, 0)$ and the plane $x + 3y + 5z = 0$

6. Find the area of the triangle with vertices $A = (1, 1, 1)$, $B = (2, 4, 6)$ and $C = (3, 5, 10)$.

7. Find a condition on a, b and c so that the system below is consistent:

$$\begin{aligned} x + 5y + z &= a \\ 2x + y + 3z &= b \\ 7x - 19y + 13z &= c \end{aligned}$$

8. How many solutions does the following system have?

$$\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$$

9. Find the point of intersection of the following lines, if it exists:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

10. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be in $\mathbb{R}^n$. Show that all three vectors are in $\text{span}(\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w})$

11. Find the general equation of the plane spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

12. Describe the span of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

13. Find the inverse of BB^T for $B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -3 & 4 \end{bmatrix}$

14. The following sequence of row operations turns A into I. Write A as a product of elementary matrices.
$R_2 \leftrightarrow R_3, \qquad \frac{1}{2}R_3, \qquad R_1 \to R_1 - 3R_3$

15. Find the LU factorization of A and use it to solve the system below:

$$\begin{aligned} 3x + \ 5y + \quad\ &= 23 \\ 6x - \ 4y + 2z &= -8 \\ 3x - \ 2y + \quad\ &= -5 \end{aligned}$$

16. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ has RREF= $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 1 & 5 \\ 4 & 8 & 1 & 6 \end{bmatrix}$ has RREF= $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Find a basis for:
a) row(A)
b) row(A) consisting of rows of A
c) col(A)
d) null(A)

17. If A is 4×6, list the possible values of nullity(A).

18. Consider the following linear transformations:
S: $\mathbb{R}^2 \to \mathbb{R}^2$ is a reflection in the x-axis
T: $\mathbb{R}^2 \to \mathbb{R}^2$ is a (counterclockwise) rotation by $\frac{\pi}{3}$
Find $S(T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}))$

19. T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.
Given $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $T(\begin{bmatrix} 6 \\ 8 \end{bmatrix})$.

20. Show that T is not linear:
$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} xy + z \\ x + y + z \end{bmatrix}$$

21. Find y:
$$\begin{aligned} 3x - y + 4z &= 32 \\ 5x - 2y - z &= 39 \\ 6x + 2y + z &= 38 \end{aligned}$$

22. Do $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

23. Consider $A = \begin{bmatrix} 4 & -2 & -2 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}$
a) Find the eigenvalues of A
b) Consider the eigenvalue with the largest algebraic multiplicity. Find a basis for the eigenspace corresponding to this eigenvalue.

24. A is a 2×2 matrix with eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -1$ corresponding to eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Find $A^{-5} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

25. $P = \begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix}$ diagonalizes A to produce $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$. Find A^7.

26. Find the coordinate vector $[\mathbf{v}]_\beta$ of $\mathbf{v} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$ with respect to the orthogonal basis $\beta = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \}$ of $\mathbb{R}^3$.

27. Find the orthogonal decomposition of $\mathbf{u} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$ with respect to W=span$\left(\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right)$.

28. Find a basis for $W^\perp$ where W is the line $2x - 5y = 0$ in $\mathbb{R}^2$.

29. Let W =span$\left(\begin{bmatrix} 2 \\ 3 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right)$. Find a basis for $W^\perp$.

30. Find an orthogonal basis for span$\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$.

31. Use the spectral decomposition to find a matrix A with eigenvalue 2 corresponding to eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and eigenvalue -2 corresponding to eigenvector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

32. Find the least squares solution to the following system:

$$\begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

33. Find the least squares regression line for the following set of points: $(1,2), (2,3), (3,5)$ and $(4,7)$.

34. Express $z = \frac{3-7i}{c+di}$ in rectangular form.

35. Express $z = 5 + 12i$ in polar form.

36. Find all cube roots of -8