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$AB = AC$

$A(B-C) = \mathbf{0} \leftarrow$ zero matrix

Let $B - C = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$A(B-C) = \mathbf{0} \Rightarrow \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3w+y & 3x+z \\ 12w+4y & 12x+4z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Choose any solution $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Say $w = 1, y = -3, x = 0, z = 0$

$$B - C = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}$$

Many possibilities for $B$ and $C$:

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$

etc.

(26) Find a nontrivial Solution to

$(A-3I)\vec{x} = \vec{0}$

$\left[ A-3I \mid \vec{0} \right]$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & 2 & -2 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - 4R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$R_2/(-2) \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$ $\quad x_2 - x_3 = 0$ $\quad x_1 = 0$

$x_2 = t$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t \quad (t \neq 0)$

For example, $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

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$$\text{Volume} = \left| \det \begin{bmatrix} 2 & 1 & 9 \\ 4 & -3 & 1 \\ 5 & -5 & 2 \end{bmatrix} \right|$$

absolute value

$$= \left| 2\begin{vmatrix} -3 & 1 \\ -5 & 2 \end{vmatrix} - \begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} + 9\begin{vmatrix} 4 & -3 \\ 5 & -5 \end{vmatrix} \right|$$

absolute value

$$= \left| 2(-1) - 3 + 9(-5) \right|$$

$$= |-50|$$

$$= 50$$

(28) If the line has equation $y = a + bx$ then:

$$\underbrace{\begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \end{array}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{array}{c} y \\ \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \end{array}}_{\vec{b}}$$

Least-squares solution $x^* = (A^T A)^{-1} A^T \vec{b}$

$$= \left( \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -5 & 0 & 5 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 \\ 10 & 150 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 55 \end{bmatrix}$$

$$= \frac{1}{500} \begin{bmatrix} 150 & -10 \\ -10 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 55 \end{bmatrix}$$

$$= \frac{1}{500} \begin{bmatrix} 350 \\ 160 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{10} \\ \frac{8}{25} \end{bmatrix}$$

$$\boxed{y = \frac{7}{10} + \frac{8}{25} x}$$

(29) $|A-\lambda I| = \begin{vmatrix} 7-\lambda & -1 \\ 5 & 5-\lambda \end{vmatrix}$

$= (7-\lambda)(5-\lambda) + 5$

$= 35 - 12\lambda + \lambda^2 + 5$

$= \lambda^2 - 12\lambda + 40$

Set $\lambda^2 - 12\lambda + 40 = 0$

$\lambda = \dfrac{12 \pm \sqrt{-16}}{2}$

$\lambda = \dfrac{12 \pm 4i}{2}$

$\boxed{\lambda = 6 \pm 2i}$

$E_{6+2i}$ : $[A - (6+2i)I \mid \vec{0}]$

$\left[\begin{array}{cc|c} 1-2i & -1 & 0 \\ 5 & -1-2i & 0 \end{array}\right]$

$R_1/(1-2i) = R_1 \cdot \frac{(1+2i)}{5}$ $\left[\begin{array}{cc|c} 1 & -\frac{1}{5}(1+2i) & 0 \\ 5 & -1-2i & 0 \end{array}\right]$

$R_2 - 5R_1$ $\left[\begin{array}{cc|c} 1 & -\frac{1}{5}(1+2i) & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t \qquad x_1 = \frac{1}{5}(1+2i)\,t$

$$\vec{x} = \begin{bmatrix} \frac{1+2i}{5} \\ 1 \end{bmatrix} t \qquad (t \neq 0)$$

$$\boxed{\text{Basis for } E_{6+2i} : \left\{ \begin{bmatrix} \frac{1+2i}{5} \\ 1 \end{bmatrix} \right\}}$$

$E_{6-2i}$ : $\quad [A-(6-2i)I \mid \vec{0}\,]$

$$\left[\begin{array}{cc|c} 1+2i & -1 & 0 \\ 5 & -1+2i & 0 \end{array}\right]$$

$$R_1/(1+2i) = R_1 \cdot \frac{(1-2i)}{5} \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{5}(1-2i) & 0 \\ 5 & -1+2i & 0 \end{array}\right]$$

$$R_2 - 5R_1 \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{5}(1-2i) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t \qquad x_1 = \frac{1}{5}(1-2i)\,t$

$$\vec{x} = \begin{bmatrix} \frac{1-2i}{5} \\ 1 \end{bmatrix} t \quad (t \neq 0)$$

$$\boxed{\text{Basis for } E_{6-2i} : \left\{ \begin{bmatrix} \frac{1-2i}{5} \\ 1 \end{bmatrix} \right\}}$$

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$$w\, CO_2 + x\, H_2O \rightarrow y\, C_6H_{12}O_6 + z\, O_2$$

C: $w = 6y \Rightarrow w - 6y = 0$

O: $2w + x = 6y + 2z \Rightarrow 2w + x - 6y - 2z = 0$

H: $2x = 12y \Rightarrow 2x - 12y = 0$

$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{array}\right]$$

(31)

$$A - 4B = \begin{bmatrix} -2 & 3 \\ 8 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & -5 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 \\ 8 & 5 \end{bmatrix} + \begin{bmatrix} -4 & 20 \\ -20 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 23 \\ -12 & 1 \end{bmatrix}$$

$$(A-4B)^T = \begin{bmatrix} -6 & -12 \\ 23 & 1 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$(A-4B)^T C^2 = \begin{bmatrix} -6 & -12 \\ 23 & 1 \end{bmatrix}\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} -222 & -324 \\ 176 & 252 \end{bmatrix}$$

(32)

Let $c_1\begin{bmatrix}1&0\\2&0\end{bmatrix}+c_2\begin{bmatrix}3&0\\6&7\end{bmatrix}+c_3\begin{bmatrix}4&3\\8&5\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1&3&4&0\\0&0&3&0\\2&6&8&0\\0&7&5&0 \end{array}\right] \end{array}$$

$$R_3-2R_1 \quad \left[\begin{array}{ccc|c} 1&3&4&0\\0&0&3&0\\0&0&0&0\\0&7&5&0 \end{array}\right]$$

$$\text{Reorder Rows} \quad \left[\begin{array}{ccc|c} 1&3&4&0\\0&7&5&0\\0&0&3&0\\0&0&0&0 \end{array}\right] \text{REF}$$

The only solution is

$c_1=0$
$c_2=0$
$c_3=0$

Yes, the matrices are linearly independent

(33)

Let $c_1\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix}+c_2\begin{bmatrix}3 & 0\\ 6 & 7\end{bmatrix}+c_3\begin{bmatrix}4 & 3\\ 8 & 5\end{bmatrix}=\begin{bmatrix}w & x\\ y & z\end{bmatrix}$

Each zero row of the REF will lead to a condition on $w, x, y, z$.

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & w\\ 0 & 0 & 3 & x\\ 2 & 6 & 8 & y\\ 0 & 7 & 5 & z \end{array}\right]$$

$$R_3-2R_1\quad \left[\begin{array}{ccc|c} 1 & 3 & 4 & w\\ 0 & 0 & 3 & x\\ 0 & 0 & 0 & y-2w\\ 0 & 7 & 5 & z \end{array}\right]$$

$$\text{Reorder Rows}\quad \left[\begin{array}{ccc|c} 1 & 3 & 4 & w\\ 0 & 7 & 5 & z\\ 0 & 0 & 3 & x\\ 0 & 0 & 0 & y-2w \end{array}\right]\ \text{REF}$$

System is consistent

$\Rightarrow\ y-2w=0$

$\Rightarrow\ y=2w$

The span is $\left\{\begin{bmatrix}w & x\\ y & z\end{bmatrix}\right.$ such that $\left.y=2w\right\}$.

The span is $\left\{\begin{bmatrix}w & x\\ 2w & z\end{bmatrix}\right\}$.