(1) Recall $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}$ for any $\vec{w}$ in $\mathbb{R}^n$

Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^n$.

$$\frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2$$

$$= \frac{1}{4}(\vec{u}+\vec{v})\cdot(u+\vec{v}) - \frac{1}{4}(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \frac{1}{4}\left[\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}\right] - \frac{1}{4}\left[\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}\right]$$

$$= \frac{1}{4}\left[\vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}\right] - \frac{1}{4}\left[\vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}\right]$$

since $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

$$= \frac{1}{4}\vec{u}\cdot\vec{u} + \frac{1}{2}\vec{u}\cdot\vec{v} + \frac{1}{4}\vec{v}\cdot\vec{v} - \frac{1}{4}\vec{u}\cdot\vec{u} + \frac{1}{2}\vec{u}\cdot\vec{v} - \frac{1}{4}\vec{v}\cdot\vec{v}$$

$$= \vec{u}\cdot\vec{v}$$

(2) Choose 3 points P, Q, R on the plane so that $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are not scalar multiples.

$P = (2,0,0)$ $\quad Q = (0,5,0)$ $\quad R = (0,0,-5/2)$

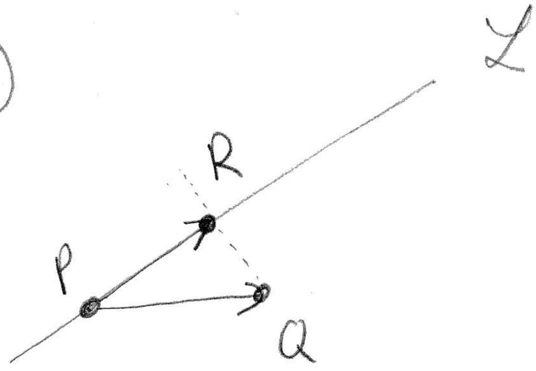
$$\left.\begin{array}{l} \vec{u} = \overrightarrow{PQ} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} \\ \\ \vec{v} = \overrightarrow{PR} = \begin{bmatrix} -2 \\ 0 \\ -5/2 \end{bmatrix} \end{array}\right] \text{ not scalar multiples } \checkmark$$

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -5/2 \end{bmatrix}$$

Many possible answers all describe the same plane.

(3)



$\mathcal{L}$ has direction

$$\vec{d} = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

$Q = (2, 1, 9)$

$P = (1, 5, 6)$ ← Choose any point on $\mathcal{L}$

$$\overrightarrow{PR} = \operatorname{proj}_{\vec{d}} \overrightarrow{PQ} \qquad \overrightarrow{PQ} = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{\vec{d} \cdot \overrightarrow{PQ}}{\|\vec{d}\|^2} \vec{d}$$

$$= \frac{9}{25} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

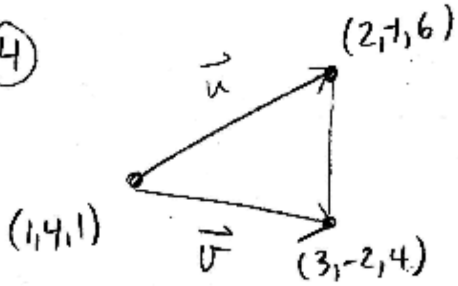
Now $\vec{R} = \vec{P} + \overrightarrow{PR}$

$$= [1, 5, 6] + \frac{9}{25}[-3, 0, 4]$$

$$= \left[\frac{-2}{25}, 5, \frac{186}{25}\right]$$

So $R = \left(\frac{-2}{25}, 5, \frac{186}{25}\right)$

4



$$\vec{u} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

$$\text{Area}(A) = \frac{1}{2} \| \vec{u} \times \vec{v} \|$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} 15 \\ 7 \\ 4 \end{bmatrix}$$

$$\begin{array}{ccccc} 1 & -5 & 5 & 1 & -5 \\ 2 & -6 & 3 & 2 & -6 \end{array}$$

$$\| \vec{u} \times \vec{v} \| = \sqrt{15^2 + 7^2 + 4^2}$$

$$= \sqrt{290}$$

$$\text{Area}(A) = \frac{\sqrt{290}}{2}$$

(5) Solve $\begin{bmatrix}1\\-6\\5\end{bmatrix} + s\begin{bmatrix}-1\\1\\4\end{bmatrix} = \begin{bmatrix}3\\-2\\3\end{bmatrix} + t\begin{bmatrix}-4\\1\\13\end{bmatrix}$

$$s\begin{bmatrix}-1\\1\\4\end{bmatrix} - t\begin{bmatrix}-4\\1\\13\end{bmatrix} = \begin{bmatrix}2\\4\\-2\end{bmatrix}$$

$$s\begin{bmatrix}-1\\1\\4\end{bmatrix} + t\begin{bmatrix}4\\-1\\-13\end{bmatrix} = \begin{bmatrix}2\\4\\-2\end{bmatrix}$$

$$\begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \left[\begin{array}{cc|c} -1 & 4 & 2\\ 1 & -1 & 4\\ 4 & -13 & -2 \end{array}\right] \end{array}$$

$$R_1/(-1) \quad \left[\begin{array}{cc|c} 1 & -4 & -2\\ 1 & -1 & 4\\ 4 & -13 & -2 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-R_1 \\ R_3-4R_1 \end{array} \left[\begin{array}{cc|c} 1 & -4 & -2\\ 0 & 3 & 6\\ 0 & 3 & 6 \end{array}\right]$$

$$R_2/3 \quad \left[\begin{array}{cc|c} 1 & -4 & -2\\ 0 & 1 & 2\\ 0 & 3 & 6 \end{array}\right]$$

$$\begin{array}{l} R_1+4R_2 \\ \\ R_3-3R_2 \end{array} \begin{array}{c} \begin{array}{cc} s & t \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 6\\ 0 & 1 & 2\\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$s = 6$

$t = 2$

$$\vec{x} = \begin{bmatrix}1\\-6\\5\end{bmatrix} + 6\begin{bmatrix}-1\\1\\4\end{bmatrix} = \begin{bmatrix}-5\\0\\29\end{bmatrix}$$

Check: $\vec{x} = \begin{bmatrix}3\\-2\\3\end{bmatrix} + 2\begin{bmatrix}-4\\1\\13\end{bmatrix}$ ✓

(6)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 1 & 14 \\ 2 & 2 & 3 & 3 & 9 \\ 7 & 8 & 10 & 8 & 39 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-R_1 \\ R_3-2R_1 \\ R_4-7R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 0 & 10 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 11 \end{array}\right]$$

$$\begin{array}{l} R_1-R_2 \\ \\ \\ R_4-R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -6 \\ 0 & 1 & 2 & 0 & 10 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1+R_3 \\ R_2-2R_3 \\ \\ R_4-R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -2 & 8 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$\boxed{d=t}$

$c+d=1$

$c=1-d$ $\quad \boxed{c=1-t}$

$b-2d=8$

$b=8+2d$ $\quad \boxed{b=8+2t}$

$a+2d=-5$

$a=-5-2d$ $\quad \boxed{a=-5-2t}$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

7

Show that $x\begin{bmatrix}-1\\3\end{bmatrix} + y\begin{bmatrix}-4\\15\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

has a solution.

$$\begin{array}{cc} x & y \\ \end{array}$$

$$\left[\begin{array}{cc|c} -1 & -4 & a \\ 3 & 15 & b \end{array}\right]$$

$R_1/(-1)$ $$\left[\begin{array}{cc|c} 1 & 4 & -a \\ 3 & 15 & b \end{array}\right]$$

R_2-3R_1 $$\left[\begin{array}{cc|c} 1 & 4 & -a \\ 0 & 3 & b+3a \end{array}\right]$$ REF

System is consistent (solvable).

Therefore $\text{span}\left(\begin{bmatrix}-1\\3\end{bmatrix}, \begin{bmatrix}-4\\15\end{bmatrix}\right) = \mathbb{R}^2$

(8) Solve $c_1\begin{bmatrix}1\\3\\10\end{bmatrix}+c_2\begin{bmatrix}-2\\1\\1\end{bmatrix}+c_3\begin{bmatrix}-1\\2\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0\\ 3 & 1 & 2 & 0\\ 10 & 1 & 5 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ R_2-3R_1 \\ R_3-10R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0\\ 0 & 7 & 5 & 0\\ 0 & 21 & 15 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ R_2/7 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 0\\ 0 & 1 & 5/7 & 0\\ 0 & 21 & 15 & 0 \end{array}\right]$$

$$\begin{array}{c} R_1+2R_2 \\ \\ R_3-21R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3/7 & 0\\ 0 & 1 & 5/7 & 0\\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$c_3=t$

$c_2+\frac{5}{7}c_3=0$

$c_2=-\frac{5}{7}c_3$ $\boxed{c_2=-\frac{5}{7}t}$

$c_1+\frac{3}{7}c_3=0$

$c_1=-\frac{3}{7}c_3$ $\boxed{c_1=-\frac{3}{7}t}$

Let $t=1$:

$c_1=\frac{-3}{7}$ $c_2=-\frac{5}{7}$ $c_3=1$

$$-\frac{3}{7}\begin{bmatrix}1\\3\\10\end{bmatrix}-\frac{5}{7}\begin{bmatrix}-2\\1\\1\end{bmatrix}+\begin{bmatrix}-1\\2\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1\\2\\5\end{bmatrix}=\frac{3}{7}\begin{bmatrix}1\\3\\10\end{bmatrix}+\frac{5}{7}\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

[Two other possibilities].

(9) Want to show that

$$C^T = -C$$

$$\begin{aligned} C^T &= (B - B^T)^T \\ &= B^T - (B^T)^T \\ &= B^T - B \\ &= -(B - B^T) \\ &= -C \end{aligned}$$

(10) a)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 8 & -6 & -4 & 0 & 1 & 0 \\ -4 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-8R_1 \\ R_3+4R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -22 & -44 & -8 & 1 & 0 \\ 0 & 11 & 21 & 4 & 0 & 1 \end{array}\right]$$

$$R_2/(-22) \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 8/22 & -1/22 & 0 \\ 0 & 11 & 21 & 4 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1-2R_2 \\ \\ R_3-11R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{6}{22} & 2/22 & 0 \\ 0 & 1 & 2 & 8/22 & -1/22 & 0 \\ 0 & 0 & -1 & 0 & \frac{11}{22} & 1 \end{array}\right] \quad \begin{array}{l} 1-\frac{16}{22}=\frac{6}{22} \\ \\ 4-\frac{88}{22}=0 \end{array}$$

$$R_3/(-1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 6/22 & 2/22 & 0 \\ 0 & 1 & 2 & 8/22 & -1/22 & 0 \\ 0 & 0 & 1 & 0 & -\frac{11}{22} & -1 \end{array}\right]$$

$$\begin{array}{l} R_1-R_3 \\ R_2-2R_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6/22 & 13/22 & 1 \\ 0 & 1 & 0 & 8/22 & 21/22 & 2 \\ 0 & 0 & 1 & 0 & -\frac{11}{22} & -1 \end{array}\right]$$

A^{-1}

or $A^{-1} = \frac{1}{22}\begin{bmatrix} 6 & 13 & 22 \\ 8 & 21 & 44 \\ 0 & -11 & -22 \end{bmatrix}$

b) $A\vec{x} = \vec{b}$

$$\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 6 & 13 & 22 \\ 8 & 21 & 44 \\ 0 & -11 & -22 \end{bmatrix}\begin{bmatrix} 5 \\ -70 \\ 33 \end{bmatrix}$$

$$= \frac{1}{22}\begin{bmatrix} -154 \\ 22 \\ 44 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 1 \\ 2 \end{bmatrix}$$

11. a)

$$\begin{array}{l} \\ R_2+4R_1 \\ R_3-3R_1 \\ R_4-2R_1 \end{array}\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 3 & -2 & 7 \\ 0 & -2 & 0 & -2 \end{bmatrix}\begin{array}{l} \\ k=-4 \\ k=3 \\ k=2 \end{array}$$

$$\begin{array}{l} \\ \\ R_3-3R_2 \\ R_4+2R_2 \end{array}\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 8 & 12 \end{bmatrix}\begin{array}{l} \\ \\ k=3 \\ k=-2 \end{array}$$

$$\begin{array}{l} \\ \\ \\ R_4+\frac{8}{14}R_3 \end{array}\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 0 & 4 \end{bmatrix}\begin{array}{l} \\ \\ \\ k=-\frac{8}{14}=-\frac{4}{7} \end{array}$$

$$A=\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 2 & -2 & -\frac{4}{7} & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 0 & 4 \end{bmatrix}}_{U}$$

b) $A\vec{x} = \vec{b}$

$LU\vec{x} = \vec{b}$

Solve $L\vec{y} = \vec{b}$, then $U\vec{x} = \vec{y}$

$L\vec{y}=\vec{b}$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 2 & -2 & -\frac{4}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 19 \\ 30 \\ -12 \end{bmatrix}$

$\boxed{y_1 = 1}$

$-4y_1 + y_2 = 19$

$-4 + y_2 = 19$

$\boxed{y_2 = 23}$

$3y_1 + 3y_2 + y_3 = 30$

$3 + 69 + y_3 = 30$

$\boxed{y_3 = -42}$

$2y_1 - 2y_2 - \frac{4}{7}y_3 + y_4 = -12$

$2 - 46 + 24 + y_4 = -12$

$\boxed{y_4 = 8}$

$U\vec{x}=\vec{y}$: $\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 23 \\ -42 \\ 8 \end{bmatrix}$

$4x_4 = 8$ $\quad x_4 = 2$

$-14x_3 - 14x_4 = -42$ $\quad x_3 = 1$

$x_2 + 4x_3 + 7x_4 = 23$ $\quad x_2 = 5$

$2x_1 + x_3 + x_4 = 1$ $\quad x_1 = -1$

$\vec{x} = \begin{bmatrix} -1 \\ 5 \\ 1 \\ 2 \end{bmatrix}$

(12) $S = \left\{ \begin{bmatrix} 4y \\ y \\ -7y \end{bmatrix} \text{ where } y \text{ is a real \#} \right\}$

$= \text{Span}\left(\begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix} \right)$

1) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in S ✓

2) Let $\vec{u}$ and $\vec{v}$ be in S.

$\vec{u} = \begin{bmatrix} 4a \\ a \\ -7a \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 4b \\ b \\ -7b \end{bmatrix}$

$\vec{u} + \vec{v} = \begin{bmatrix} 4a + 4b \\ a+b \\ -7a - 7b \end{bmatrix}$

$= \begin{bmatrix} 4(a+b) \\ a+b \\ -7(a+b) \end{bmatrix}$

$\vec{u} + \vec{v}$ is in S.

3) Let $\vec{u} = \begin{bmatrix} 4a \\ a \\ -7a \end{bmatrix}$ be in S.

$c\vec{u} = \begin{bmatrix} 4ac \\ ac \\ -7ac \end{bmatrix} = \begin{bmatrix} 4(ac) \\ ac \\ -7(ac) \end{bmatrix}$

$c\vec{u}$ is in S.

(13) Form A^T because columns don't move around. Find a basis for $col(A^T)$.

$$A^T = \begin{bmatrix} 1 & -2 & -1 & 4 \\ 2 & -3 & 1 & 3 \\ -1 & 1 & -2 & 3 \\ 9 & 6 & 63 & 3 \end{bmatrix}$$

Locate leading entries in REF.

$$\begin{array}{l} \\ R_2-2R_1 \\ R_3+R_1 \\ R_4-9R_1 \end{array} \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & 3 & -5 \\ 0 & -1 & -3 & 7 \\ 0 & 24 & 72 & -33 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ R_3+R_2 \\ R_4-24R_2 \end{array} \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 87 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ R_3/2 \\ R_4-\frac{87}{2}R_3 \end{array} \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & 3 & -5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading entries in columns 1, 2, 4

$\Rightarrow$ Columns 1, 2, 4 of A^T form a basis for $col(A^T)$.

$\Rightarrow$ Rows 1, 2, 4 of A form a basis for $row(A)$

$$\{[1 \;\; 2 \;\; -1 \;\; 9], [-2 \;\; -3 \;\; 1 \;\; 6], [4 \;\; 3 \;\; 3 \;\; 3]\}$$