Math 251 Practice Problems

1. Prove that $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}||\mathbf{u} + \mathbf{v}||^2 - \frac{1}{4}||\mathbf{u} - \mathbf{v}||^2$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

2. Find the vector form of the plane $5x + 2y - 4z = 10$.

3. Line $\mathcal{L}$ is given by $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$. Find the point R on $\mathcal{L}$ that is closest to $Q = (2, 1, 9)$.

4. Find the area of the triangle with vertices $(1, 4, 1), (2, -1, 6)$ and $(3, -2, 4)$.

5. Find the point of intersection of the lines given by
$\mathbf{x} = \begin{bmatrix} 1 \\ -6 \\ 5 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 13 \end{bmatrix}$,
or show that there is no intersection.

6. Solve using either Gaussian or Gauss-Jordan Elimination.

$$\begin{aligned} a + b + c + d &= 4 \\ a + 2b + 3c + d &= 14 \\ 2a + 2b + 3c + 3d &= 9 \\ 7a + 8b + 10c + 8d &= 39 \end{aligned}$$

7. Show that the span of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 15 \end{bmatrix}$ is all of $\mathcal{R}^2$.

8. Write one of the vectors below as a linear combination of the other two:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}.$$

9. A square matrix A is called **skew symmetric** if $A^T = -A$. Show that if B is an $n \times n$ matrix then $C = B - B^T$ is skew symmetric.

10. a) Find the inverse of the matrix below using Gauss-Jordan Elimination.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 8 & -6 & -4 \\ -4 & 3 & 1 \end{bmatrix}$$

b) Use part a) to solve the system below.

$$\begin{aligned} x + 2y + 5z &= 5 \\ 8x - 6y - 4z &= -70 \\ -4x + 3y + z &= 33 \end{aligned}$$

11. a) Find the LU factorization of the matrix below.

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ -8 & 1 & 0 & 3 \\ 6 & 3 & 1 & 10 \\ 4 & -2 & 2 & 0 \end{bmatrix}$$

b) Use part a) to solve the system below.

$$\begin{aligned} 2x_1 \quad\quad + x_3 + x_4 &= 1 \\ -8x_1 + x_2 \quad\quad + 3x_4 &= 19 \\ 6x_1 + 3x_2 + x_3 + 10x_4 &= 30 \\ 4x_1 - 2x_2 + 2x_3 \quad\quad &= -12 \end{aligned}$$

12. Show that the set S of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x = 4y$ and $z = -7y$ is a subspace of $\mathbb{R}^3$.

13. Find a basis for the row space of A consisting of rows of A.

$$A = \begin{bmatrix} 1 & 2 & -1 & 9 \\ -2 & -3 & 1 & 6 \\ -1 & 1 & -2 & 63 \\ 4 & 3 & 3 & 3 \end{bmatrix}$$

14. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 12 \\ 13 \\ -6 \end{bmatrix}$.

Find $T(\mathbf{v})$ for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ with:

$$T(\mathbf{v}_1) = \begin{bmatrix} -6 \\ 30 \\ 24 \\ 8 \end{bmatrix}, T(\mathbf{v}_2) = \begin{bmatrix} -2 \\ 8 \\ 6 \\ 2 \end{bmatrix} \text{ and } T(\mathbf{v}_3) = \begin{bmatrix} -1 \\ 9 \\ 6 \\ 0 \end{bmatrix}.$$

15. Find $\det A$ using Gaussian Elimination.

$$A = \begin{bmatrix} 2 & 1 & 6 & 9 \\ 2 & 1 & 3 & 0 \\ 6 & 8 & 1 & 0 \\ 4 & 12 & 1 & 1 \end{bmatrix}$$

16. Use the adjoint formula to find A^{-1}.

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \end{bmatrix}$$

17. Given $A = \begin{bmatrix} 2 & 1 & 1 & 6 \\ 0 & 4 & 2 & 0 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

a) Find the characteristic polynomial of A.
b) Find a basis for the eigenspace E_2.
c) Explain why A is not diagonalizable.

18. Find the matrix P that diagonalizes $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix}$.

19. $\mathcal{B} = \{\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} -69 \\ 45 \\ 42 \end{bmatrix}\}$ is an orthogonal basis for $\mathbb{R}^3$.

Find a vector $\mathbf{u}$ in $\mathbb{R}^3$ such that:

$$\mathbf{u} \cdot \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} = -150, \quad \mathbf{u} \cdot \begin{bmatrix} 7 \\ 7 \\ 4 \end{bmatrix} = 342 \quad \text{and} \quad \mathbf{u} \cdot \begin{bmatrix} -69 \\ 45 \\ 42 \end{bmatrix} = 2850.$$

20. Let $W = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x = 5t, y = -2t, z = 10t\}$.

Find a basis for $W^{\perp}$.

21. a) Find an orthonormal basis for span $(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix})$.

b) Use part a) to find the QR factorization of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

22. Find the matrix Q that orthogonally diagonalizes $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$.

Hint: The characteristic polynomial of A is $(\lambda - 2)(\lambda - 5)^2$.

23. Express $z = \frac{x+yi}{13+5i}$ in the form $a + bi$.

24. Find all the cube roots of $-8i$. Express each root in the form $a + bi$.

25. Let $A = \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}$. Find 2×2 matrices B and C such that $AB = AC$ but $B \neq C$.

26. Find one eigenvector corresponding to $\lambda = 3$ for the matrix
$A = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$.

27. Find the volume of the parallelepiped determined by $\mathbf{a} = [2, 1, 9]$, $\mathbf{b} = [4, -3, 1]$ and $\mathbf{c} = [5, -5, 2]$.

28. Find the least-squares approximating line (also called the least-squares regression line) for the following set of points: $(-5, -1), (0, 1), (5, 2), (10, 4)$.

29. Find any real or complex eigenvalues of the matrix $A = \begin{bmatrix} 7 & -1 \\ 5 & 5 \end{bmatrix}$, and find a basis for each eigenspace.