1. [3 marks] Solve:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 3 & 1 & 5 \\ 0 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{Call this } \bar{y}} = \begin{bmatrix} 38 \\ 63 \\ 96 \end{bmatrix}$$

1) Solve $L\bar{y} = \bar{b}$

$$\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 38 \\ 2 & 1 & 0 & 63 \\ 2 & -2 & 1 & 96 \end{array}\right] \end{array}$$

$y_1 = 38$

$2y_1 + y_2 = 63 \to 76 + y_2 = 63 \to y_2 = -13$

$2y_1 - 2y_2 + y_3 = 96 \to 76 + 26 + y_3 = 96 \to y_3 = -6$

2) Solve $U\bar{x} = \bar{y}$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 3 & 1 & 5 & 38 \\ 0 & 4 & -3 & -13 \\ 0 & 0 & -2 & -6 \end{array}\right] \end{array}$$

$-2x_3 = -6 \to x_3 = 3$

$4x_2 - 3x_3 = -13 \to 4x_2 - 9 = -13 \to x_2 = -1$

$3x_1 + x_2 + 5x_3 = 38 \to 3x_1 - 1 + 15 = 38 \to x_1 = 8$

$$\bar{x} = \begin{bmatrix} 8 \\ -1 \\ 3 \end{bmatrix}$$

2. [3 marks] Use Cramer's Rule to find y:

$$\begin{aligned} -y + z &= -5 \\ 4x + 3y + 7z &= 33 \\ 2x + y - 8z &= 2 \end{aligned}$$

$$|A| = \begin{vmatrix} 0 & -1 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & -8 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & 7 \\ 2 & -8 \end{vmatrix} + 1\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= -46 - 2$$

$$= -48$$

$$|A_2| = \begin{vmatrix} 0 & -5 & 1 \\ 4 & 33 & 7 \\ 2 & 2 & -8 \end{vmatrix}$$

$$= 5\begin{vmatrix} 4 & 7 \\ 2 & -8 \end{vmatrix} + 1\begin{vmatrix} 4 & 33 \\ 2 & 2 \end{vmatrix}$$

$$= 5(-46) - 58$$

$$= -288$$

$$y = \frac{|A_2|}{|A|} = 6$$

3. [3 marks] Find all eigenvectors of $A = \begin{bmatrix} 6 & 5 \\ -8 & -8 \end{bmatrix}$ corresponding to $\lambda = 2$

Solve $[A - \lambda I \mid \bar{0}]$

$$\left[\begin{array}{cc|c} 4 & 5 & 0 \\ -8 & -10 & 0 \end{array}\right]$$

$\frac{R_1}{4}$ $$\left[\begin{array}{cc|c} 1 & \frac{5}{4} & 0 \\ -8 & -10 & 0 \end{array}\right]$$

$R_2 + 8R_1$ $$\left[\begin{array}{cc|c} 1 & \frac{5}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$x_2 = t$

$$x_1 + \frac{5}{4}t = 0 \rightarrow x_1 = -\frac{5}{4}t$$

$$\bar{x} = \begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix} t \quad (t \neq 0)$$

or $$\bar{x} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} t \quad (t \neq 0)$$

4. [3 marks] The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates a vector by $120°$ counter-clockwise then performs the transformation S below. Find the standard matrix for T.

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 8y \\ 6x \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{\theta = 120°}$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$[S] = \begin{bmatrix} 2 & -8 \\ 6 & 0 \end{bmatrix}$$

$$[T] = [S][T_1]$$

$$= \begin{bmatrix} 2 & -8 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 - 4\sqrt{3} & -\sqrt{3} + 4 \\ -3 & -3\sqrt{3} \end{bmatrix}$$

5. [3 marks] $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 1 \end{bmatrix}$ has RREF$= \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Find a basis for:

a) the column space of A

1st and 3rd columns of A

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}$

b) the row space of A

nonzero rows of RREF

$\{ [1 \ 3 \ 0], [0 \ 0 \ 1] \}$

c) the null space of A

$A\bar{x} = \bar{0}$

$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

$\uparrow$

$x_2 = t$

$x_1 + 3t = 0 \rightarrow x_1 = -3t$

$x_3 = 0$

$\bar{x} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} t$

Basis $= \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$

6. [2 marks] A is a 5×8 matrix.

a) What are the possible values of rank(A)?

rank = # pivots in REF/RREF

Possible values: 0, 1, 2, 3, 4, 5

b) What are the possible values of nullity(A)?

rank + nullity = # columns
= 8

Possible values: 8, 7, 6, 5, 4, 3