1. [3 marks] Solve:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -11 \\ 47 \end{bmatrix}$$

Solve $L\vec{y} = \vec{b}$

$$\begin{array}{ccc} y_1 & y_2 & y_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ -3 & 1 & 0 & -11 \\ 2 & 4 & 1 & 47 \end{array}\right]$$

$y_1 = 3$

$-3y_1 + y_2 = -11 \Rightarrow -9 + y_2 = -11 \Rightarrow y_2 = -2$

$2y_1 + 4y_2 + y_3 = 47 \Rightarrow 6 - 8 + y_3 = 47 \Rightarrow y_3 = 49$

Solve $U\vec{x} = \vec{y}$

$$\begin{array}{ccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 6 & 1 & 3 & 3 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 7 & 49 \end{array}\right]$$

$7x_3 = 49 \Rightarrow x_3 = 7$

$2x_2 - 2x_3 = -2 \Rightarrow 2x_2 - 14 = -2 \Rightarrow x_2 = 6$

$6x_1 + x_2 + 3x_3 = 3 \Rightarrow 6x_1 + 6 + 21 = 3 \Rightarrow x_1 = -4$

$$\vec{x} = \begin{bmatrix} -4 \\ 6 \\ 7 \end{bmatrix}$$

2. [3 marks] A and B are 4×4 matrices. Given that $\det(A) = a$ and $\det(B) = b$, find $\det(2AB^T)$.

$$\det(2AB^T)$$
$$= \det(2A)\det(B^T)$$
$$= 2^4 \det(A) \det(B)$$
$$= 16ab$$

3. [3 marks] Find all the eigenvalues of $A = \begin{bmatrix} 7 & -2 \\ -1 & 6 \end{bmatrix}$.

Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 7-\lambda & -2 \\ -1 & 6-\lambda \end{vmatrix} = 0$$
$$(7-\lambda)(6-\lambda) - 2 = 0$$
$$\lambda^2 - 13\lambda + 40 = 0$$
$$(\lambda - 5)(\lambda - 8) = 0$$
$$\lambda = 5, 8$$

4. [4 marks] Find all eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ corresponding to $\lambda = -2$.

Solve $\left[A - \lambda I \mid \vec{0} \right]$

$$\left[A + 2I \mid \vec{0} \right]$$

$$\left[\begin{array}{cc|c} 4 & 3 & 0 \\ 4 & 3 & 0 \end{array}\right]$$

$\frac{R_1}{4}$ $$\left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ 4 & 3 & 0 \end{array}\right]$$

$$\begin{array}{ccc} x_1 & x_2 & \\ \end{array}$$

$R_2 - 4R_1$ $$\left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_2 = t$$

$$x_1 + \frac{3}{4} x_2 = 0 \Rightarrow x_1 = -\frac{3}{4} t$$

$$\vec{x} = \begin{bmatrix} -3/4 \\ 1 \end{bmatrix} t \qquad (t \neq 0)$$

or $$\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} t \qquad (t \neq 0)$$

5. [3 marks] $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 10 & 10 \\ 4 & 8 & 13 & 18 \end{bmatrix}$ has RREF$=\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find:

a) a basis for the column space of A

Use Columns 1, 3 and 4 of A

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 10 \\ 13 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 10 \\ 18 \end{bmatrix} \right\}$$

b) dim(null(A))

= # of parameters in solution to $A\vec{x} = \vec{0}$

(or # of columns without pivots in RREF)

= 1

6. [4 marks] Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$.

T is a linear transformation such that $T(\mathbf{v}_1) = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$ and $T(\mathbf{v}_2) = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$.

Find $T(\begin{bmatrix} 109 \\ 233 \end{bmatrix})$.

Let $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} 109 \\ 233 \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 8 & 109 \\ 2 & 17 & 233 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 2R_1 \end{array} \left[\begin{array}{cc|c} 1 & 8 & 109 \\ 0 & 1 & 15 \end{array}\right]$$

$$\begin{array}{l} R_1 - 8R_2 \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & -11 \\ 0 & 1 & 15 \end{array}\right]$$

$$c_1 = -11, \quad c_2 = 15$$

$$\begin{aligned} T\left(\begin{bmatrix} 109 \\ 233 \end{bmatrix}\right) &= T(-11\vec{v}_1 + 15\vec{v}_2) \\ &= T(-11\vec{v}_1) + T(15\vec{v}_2) \quad (T \text{ is linear}) \\ &= -11\,T(\vec{v}_1) + 15\,T(\vec{v}_2) \quad (T \text{ is linear}) \\ &= -11\begin{bmatrix} 2 \\ -9 \end{bmatrix} + 15\begin{bmatrix} 4 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} 38 \\ 249 \end{bmatrix} \end{aligned}$$