1. [5 marks] Let $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ 6 \\ 3 \end{bmatrix}$.

a) Calculate $\mathbf{u} \times \mathbf{v}$.

$$\begin{matrix} 5 & 1 & -1 & 5 & 1 \\ -8 & 6 & 3 & -8 & 6 \end{matrix}$$

$$\bar{u} \times \bar{v} = [9, -7, 38]$$

1

b) Consider the triangle formed by placing **u** and **v** tail-to-tail. Find the area of the triangle.

$$\frac{1}{2} \| \bar{u} \times \bar{v} \| = \frac{1}{2} \sqrt{1574}$$

2

c) Consider the plane that has direction vectors **u** and **v** and passes through $(3, -1, 2)$. Find the general form of the plane.

$$\bar{n} \cdot \bar{x} = \bar{n} \cdot \bar{p}$$

$$\begin{bmatrix} 9 \\ -7 \\ 38 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -7 \\ 38 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$9x - 7y + 38z = 110$$

2

2. [6 marks] Let $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$. Find:

a) a vector of length one parallel to $\mathbf{u} - \mathbf{v}$

$$\vec{u} - \vec{v} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

2

$$\|\vec{u} - \vec{v}\| = \sqrt{21}$$

A vector of length one parallel to $\vec{u} - \vec{v}$ is:

$$\frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

b) the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$41 = \sqrt{62}\,\sqrt{41}\,\cos\theta$$

4

$$\theta = \cos^{-1}\left(\frac{41}{\sqrt{62}\,\sqrt{41}}\right)$$

$$\approx 36^\circ$$

3. [5 marks] Solve using Gauss-Jordan Elimination:

$$\begin{aligned} 3x + \ \ 6y + \ \ 6z &= 42 \\ 2x + \ \ \ \ y + \ \ 7z &= 19 \\ 17x + 16y + 52z &= 184 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 6 & 6 & 42 \\ 2 & 1 & 7 & 19 \\ 17 & 16 & 52 & 184 \end{array}\right]$$

$$\frac{R_1}{3} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 14 \\ 2 & 1 & 7 & 19 \\ 17 & 16 & 52 & 184 \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 17R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 14 \\ 0 & -3 & 3 & -9 \\ 0 & -18 & 18 & -54 \end{array}\right]$$

$$\frac{R_2}{-3} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 14 \\ 0 & 1 & -1 & 3 \\ 0 & -18 & 18 & -54 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ R_3 + 18R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 4 & 8 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

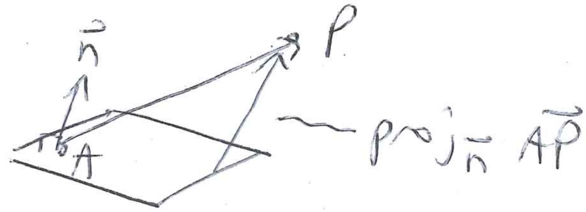
$z = t$

$x + 4z = 8 \Rightarrow x = 8 - 4t$

$y - z = 3 \Rightarrow y = 3 + t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

4. [5 marks] Find the distance between the point $P = (1, -3, 6)$ and the plane $x - 3y + 7z = 4$.



$$\text{distance} = \| \text{proj}_{\vec{n}} \vec{AP} \|$$

$A = (4, 0, 0)$ any point on plane

$$\vec{AP} = [-3, -3, 6]$$

$$\vec{n} = [1, -3, 7]$$

$$\text{proj}_{\vec{n}} \vec{AP} = \frac{\vec{n} \cdot \vec{AP}}{\|\vec{n}\|^2} \vec{n}$$

$$= \frac{48}{59} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

$$\text{distance} = \left\| \frac{48}{59} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \right\|$$

$$= \frac{48\sqrt{59}}{59}$$

5. [4 marks] How many solutions does the system below have? Your answer will depend on the value of k.

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 64 & 8 \end{array}\right]$$

$R_2 - kR_1$ $\left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 64-k^2 & 8-k \end{array}\right]$

$64-k^2 \neq 0$

$\frac{R_2}{64-k^2}$ $\left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{8-k}{64-k^2} \end{array}\right]$

1 solution

$64-k^2 = 0$
$(8-k)(8+k) = 0$

$k=8$

$\left[\begin{array}{cc|c} 1 & 8 & 1 \\ 0 & 0 & 0 \end{array}\right]$

∞-many solutions

$k=-8$

$\left[\begin{array}{cc|c} 1 & -8 & 1 \\ 0 & 0 & 16 \end{array}\right]$

no solution

No solution if $k=-8$
∞-many solutions if $k=8$
1 solution if $k \neq \pm 8$