1. [3 marks] Solve:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 0 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{call this } \bar{y}} = \begin{bmatrix} 40 \\ 64 \\ 104 \end{bmatrix}$$

1) Solve $L\bar{y} = \bar{b}$

$$\begin{array}{ccc} y_1 & y_2 & y_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 40 \\ 2 & 1 & 0 & 64 \\ 2 & -2 & 1 & 104 \end{array}\right]$$

$y_1 = 40$

$2y_1 + y_2 = 64 \rightarrow 80 + y_2 = 64 \rightarrow y_2 = -16$

$2y_1 - 2y_2 + y_3 = 104 \rightarrow 80 + 32 + y_3 = 104 \rightarrow y_3 = -8$

2) Solve $U\bar{x} = \bar{y}$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 40 \\ 0 & 4 & -3 & -16 \\ 0 & 0 & -2 & -8 \end{array}\right]$$

$-2x_3 = -8 \rightarrow x_3 = 4$

$4x_2 - 3x_3 = -16 \rightarrow 4x_2 - 12 = -16 \rightarrow x_2 = -1$

$3x_1 + x_2 + 5x_3 = 40 \rightarrow 3x_1 + 21 = 40 \rightarrow x_1 = 7$

$$\bar{x} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}$$

2. [3 marks] Use Cramer's Rule to find y:

$$\begin{aligned} -y + z &= -8 \\ 4x + 3y + 7z &= 42 \\ 2x + y - 8z &= 5 \end{aligned}$$

$$|A| = \begin{vmatrix} 0 & -1 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & -8 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 4 & 7 \\ 2 & -8 \end{vmatrix} + 1 \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= -46 - 2$$

$$= -48$$

$$|A_2| = \begin{vmatrix} 0 & -8 & 1 \\ 4 & 42 & 7 \\ 2 & 5 & -8 \end{vmatrix}$$

$$= 8 \begin{vmatrix} 4 & 7 \\ 2 & -8 \end{vmatrix} + 1 \begin{vmatrix} 4 & 42 \\ 2 & 5 \end{vmatrix}$$

$$= 8(-46) - 64$$

$$= -432$$

$$y = \frac{-432}{-48} = 9$$

3. [3 marks] Find all eigenvectors of $A = \begin{bmatrix} 6 & 3 \\ -8 & -4 \end{bmatrix}$ corresponding to $\lambda = 2$

Solve $[A - \lambda I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 4 & 3 & 0 \\ -8 & -6 & 0 \end{array}\right]$$

$$\frac{R_1}{4} \quad \left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ -8 & -6 & 0 \end{array}\right]$$

$$R_2 + 8R_1 \quad \left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_2 = t$$

$$x_1 + \frac{3}{4}t = 0 \rightarrow \quad x_1 = -\frac{3}{4}t$$

$$\vec{x} = \begin{bmatrix} -3/4 \\ 1 \end{bmatrix} t \qquad (t \neq 0)$$

or $$\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} t \qquad (t \neq 0)$$

4. [3 marks] The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first performs the transformation S below, then rotates the vector by 120° counter-clockwise. Find the standard matrix for T.

$$S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4x - 6y \\ 8x \end{bmatrix}$$

$$[S] = \begin{bmatrix} 4 & -6 \\ 8 & 0 \end{bmatrix}$$

$$[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \theta = 120°$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$[T] = [T_2][S]$$

$$= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2-4\sqrt{3} & 3 \\ 2\sqrt{3}-4 & -3\sqrt{3} \end{bmatrix}$$

5. [3 marks] $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 8 & 1 \end{bmatrix}$ has RREF$= \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Find a basis for:

a) the column space of A

1st and 3rd Colums of A

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix} \right\}$$

b) the row space of A

nonzero rows of RREF

$$\left\{ \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}$$

c) the null space of A

$$A\bar{x} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$x_2 = t$

$x_1 + 2t = 0 \rightarrow x_1 = -2t$

$x_3 = 0$

$$\bar{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

6. [2 marks]

a) A is a 3×3 matrix with rank(A)=3. Find the RREF of A.

rank = # pivots in REF/RREF

$$RREF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) B is a 3×3 matrix with nullity(B)=3. Find the RREF of B.

nullity(B) = 3 $\Rightarrow$ rank(B) = 0

$$RREF = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$