1. [3 marks] Solve:

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 \\ -41 \\ 74 \end{bmatrix}$$

Let $L\vec{y} = \vec{b}$

$$\begin{array}{cccc} y_1 & y_2 & y_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 13 \\ -3 & 1 & 0 & -41 \\ 2 & 4 & 1 & 74 \end{array}\right]$$

$y_1 = 13$

$-3y_1 + y_2 = -41 \Rightarrow -39 + y_2 = -41 \Rightarrow y_2 = -2$

$2y_1 + 4y_2 + y_3 = 74 \Rightarrow 26 - 8 + y_3 = 74 \Rightarrow y_3 = 56$

Let $U\vec{x} = \vec{y}$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 6 & 1 & 3 & 13 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 7 & 56 \end{array}\right]$$

$7x_3 = 56 \Rightarrow x_3 = 8$

$2x_2 - 2x_3 = -2 \Rightarrow 2x_2 - 16 = -2 \Rightarrow x_2 = 7$

$6x_1 + x_2 + 3x_3 = 13 \Rightarrow 6x_1 + 7 + 24 = 13 \Rightarrow x_1 = -3$

$$\vec{x} = \begin{bmatrix} -3 \\ 7 \\ 8 \end{bmatrix}$$

2. [3 marks] A and B are 4×4 matrices. The matrix B has an inverse. Given that $\det(A) = a$ and $\det(B) = b$, find $\det(3AB^{-1})$.

$$\det(3AB^{-1})$$
$$= \det(3A)\det(B^{-1})$$
$$= 3^4 \det(A) \det(B^{-1})$$
$$= 81 \det(A) \frac{1}{\det B}$$
$$= \frac{81a}{b}$$

3. [3 marks] Find all the eigenvalues of $A = \begin{bmatrix} 8 & -2 \\ -1 & 7 \end{bmatrix}$.

Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -2 \\ -1 & 7-\lambda \end{vmatrix} = 0$$
$$(8-\lambda)(7-\lambda) - 2 = 0$$
$$\lambda^2 - 15\lambda + 54 = 0$$
$$(\lambda - 6)(\lambda - 9) = 0$$
$$\lambda = 6, 9$$

4. [4 marks] Find all eigenvectors of $A = \begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$ corresponding to $\lambda = -1$.

Solve $[A - \lambda I \mid \vec{0}]$

$[A + I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 4 & 3 & 0 \\ 4 & 3 & 0 \end{array}\right]$$

$$\frac{R_1}{4} \quad \left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ 4 & 3 & 0 \end{array}\right]$$

$$R_2 - 4R_1 \quad \begin{array}{cc} x_1 & x_2 \end{array} \left[\begin{array}{cc|c} 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 + \frac{3}{4}x_2 = 0 \Rightarrow x_1 = -\frac{3}{4}t$

$$\vec{x} = \begin{bmatrix} -3/4 \\ 1 \end{bmatrix} t \quad (t \neq 0)$$

or $$\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} t \quad (t \neq 0)$$

5. [3 marks] $A = \begin{bmatrix} 4 & 8 & 13 & 18 \\ 3 & 6 & 10 & 10 \\ 2 & 4 & 6 & 8 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ has RREF= $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Find:

a) a basis for the column space of A

Use columns 1, 3 and 4 of A

$$\left\{ \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 13 \\ 10 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 18 \\ 10 \\ 8 \\ 4 \end{bmatrix} \right\}$$

b) dim(row(A))

= # vectors in a basis for row(A)

= 3

6. [4 marks] Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 8 \\ 17 \end{bmatrix}$.

T is a linear transformation such that $T(\mathbf{v}_1) = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$ and $T(\mathbf{v}_2) = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$.

Find $T(\begin{bmatrix} 92 \\ 197 \end{bmatrix})$.

Let $c_1\vec{v_1} + c_2\vec{v_2} = \begin{bmatrix} 92 \\ 197 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 8 & 92 \\ 2 & 17 & 197 \end{array}\right] \end{array}$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & 8 & 92 \\ 0 & 1 & 13 \end{array}\right]$$

$$R_1 - 8R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & -12 \\ 0 & 1 & 13 \end{array}\right]$$

$c_1 = -12$, $c_2 = 13$

$$\begin{aligned} T\left(\begin{bmatrix} 92 \\ 197 \end{bmatrix}\right) &= T(-12\vec{v_1} + 13\vec{v_2}) \\ &= T(-12\vec{v_1} + 13\vec{v_2}) && (T \text{ is linear}) \\ &= -12T(\vec{v_1}) + 13T(\vec{v_2}) && (T \text{ is linear}) \\ &= -12\begin{bmatrix} 2 \\ -9 \end{bmatrix} + 13\begin{bmatrix} 4 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} 28 \\ 238 \end{bmatrix} \end{aligned}$$