1. [5 marks] Let $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -8 \\ 6 \\ 3 \end{bmatrix}$.

a) Calculate $\mathbf{u} \times \mathbf{v}$.

$$\begin{matrix} 4 & 1 & -2 & 4 & 1 \\ -8 & 6 & 3 & -8 & 6 \end{matrix}$$

$$\vec{u} \times \vec{v} = [15, 4, 32]$$

1

b) Consider the triangle formed by placing $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail. Find the area of the triangle.

$$\frac{1}{2} \| \bar{u} \times \bar{v} \| = \frac{1}{2} \sqrt{1265}$$

2

c) Consider the plane that has direction vectors $\mathbf{u}$ and $\mathbf{v}$ and passes through $(3, -1, 2)$. Find the general form of the plane.

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

$$\begin{bmatrix} 15 \\ 4 \\ 32 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 4 \\ 32 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$15x + 4y + 32z = 105$$

2

2. [6 marks] Let $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. Find:

a) a vector of length one parallel to $\mathbf{u} - \mathbf{v}$

$$\bar{u} - \bar{v} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

2

$$\|\bar{u} - \bar{v}\| = \sqrt{29}$$

A vector of length one parallel to $\bar{u} - \bar{v}$ is:

$$\frac{1}{\sqrt{29}} \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

b) the angle between $\mathbf{u}$ and $\mathbf{v}$

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \, \|\bar{v}\| \cos\theta$$

4

$$27 = \sqrt{62}\,\sqrt{21}\,\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{27}{\sqrt{62}\,\sqrt{21}}\right)$$

$$\approx 42^\circ$$

3. [5 marks] Solve using Gauss-Jordan Elimination:

$$\begin{aligned} 3x + 6y + 6z &= 39 \\ 2x + y + 7z &= 20 \\ 17x + 16y + 52z &= 185 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 6 & 6 & 39 \\ 2 & 1 & 7 & 20 \\ 17 & 16 & 52 & 185 \end{array}\right]$$

$$\frac{R_1}{3} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 13 \\ 2 & 1 & 7 & 20 \\ 17 & 16 & 52 & 185 \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 17R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 13 \\ 0 & -3 & 3 & -6 \\ 0 & -18 & 18 & -36 \end{array}\right]$$

$$\frac{R_2}{-3} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 13 \\ 0 & 1 & -1 & 2 \\ 0 & -18 & 18 & -36 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 18R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 4 & 9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

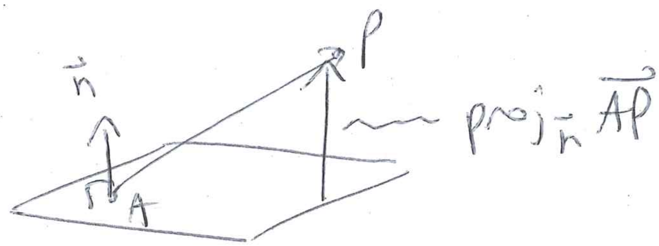
$z = t$

$x + 4z = 9 \Rightarrow x = 9 - 4t$

$y - z = 2 \Rightarrow y = 2 + t$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

4. [5 marks] Find the distance between the point $P = (1, -3, 6)$ and the plane $x - 7y + 3z = 6$.



distance $= \| proj_{\vec{n}} \vec{AP} \|$

$A = (6,0,0)$ any point on plane

$\vec{AP} = [-5, -3, 6]$ $\quad \vec{n} = [1, -7, 3]$

$$proj_{\vec{n}} \vec{AP} = \frac{\vec{n} \cdot \vec{AP}}{\|\vec{n}\|^2} \vec{n}$$

$$= \frac{34}{59} \begin{bmatrix} 1 \\ -7 \\ 3 \end{bmatrix}$$

$$\text{distance} = \left\| \frac{34}{59} \begin{bmatrix} 1 \\ -7 \\ 3 \end{bmatrix} \right\|$$

$$= \frac{34\sqrt{59}}{59}$$

5. [4 marks] How many solutions does the system below have? Your answer will depend on the value of k.

$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 81 & 9 \end{array}\right]$$

$R_2 - kR_1$ $\left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 81-k^2 & 9-k \end{array}\right]$

$81-k^2 \neq 0$

$\frac{R_2}{81-k^2}$ $\left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{9-k}{81-k^2} \end{array}\right]$

1 solution

$81-k^2=0$

$(9-k)(9+k)=0$

$k=9$

$\left[\begin{array}{cc|c} 1 & 9 & 1 \\ 0 & 0 & 0 \end{array}\right]$

∞-many solutions

$k=-9$

$\left[\begin{array}{cc|c} 1 & -9 & 1 \\ 0 & 0 & 18 \end{array}\right]$

no solution

No Solution if $k=-9$

∞-many solutions if $k=9$

1 solution if $k \neq \pm 9$