1. [4 marks] We want to balance:

$CO_2 + H_2O \rightarrow C_6H_{12}O_6 + O_2$

Set up a system of equations. DO NOT SOLVE THE SYSTEM.

$w\,CO_2 + x\,H_2O \rightarrow y\,C_6H_{12}O_6 + z\,O_2$

C: $w = 6y$

O: $2w + x = 6y + 2z$

H: $2x = 12y$

or

$$\begin{aligned} w - 6y &= 0 \\ 2w + x - 6y - 2z &= 0 \\ 2x - 12y &= 0 \end{aligned}$$

or

$$\begin{array}{cccc|c} w & x & y & z & \\ 1 & 0 & -6 & 0 & 0 \\ 2 & 1 & -6 & -2 & 0 \\ 0 & 2 & -12 & 0 & 0 \end{array}$$

2. [4 marks] A is an invertible matrix. Solve for X:
$(AX + 3I)^T = B$

$$((AX + 3I)^T)^T = B^T$$

$$AX + 3I = B^T$$

$$AX = B^T - 3I$$

$$X = A^{-1}(B^T - 3I)$$

3. [1 mark] The set of vectors $\{\mathbf{x}, \mathbf{y}\}$ is linearly dependent. Is the set of vectors $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ linearly dependent? Explain briefly.

Yes.

There's a linear dependency among $\overline{x}$ and $\overline{y}$

$\Rightarrow$ There's a linear dependency among $\overline{x}$, $\overline{y}$ and $\overline{z}$.

4. [4 marks] Given $A = \begin{bmatrix} -2 & 3 \\ 8 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -5 \\ 5 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Find $(A - 4B)^T C^2$.

$$\left(\begin{bmatrix} -2 & 3 \\ 8 & 5 \end{bmatrix} - 4 \begin{bmatrix} 1 & -5 \\ 5 & 1 \end{bmatrix} \right)^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2$$

$$= \left(\begin{bmatrix} -2 & 3 \\ 8 & 5 \end{bmatrix} + \begin{bmatrix} -4 & 20 \\ -20 & -4 \end{bmatrix} \right)^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 23 \\ -12 & 1 \end{bmatrix}^T \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -12 \\ 23 & 1 \end{bmatrix} \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} -222 & -324 \\ 176 & 252 \end{bmatrix}$$

5. [6 marks] Solve the system by finding A^{-1}.

$$\begin{aligned} x - 2y - 3z &= 16 \\ 4x - 7y - 16z &= 45 \\ -3x + 6y + 10z &= -45 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ 4 & -7 & -16 & 0 & 1 & 0 \\ -3 & 6 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2-4R_1 \\ R_3+3R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -4 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$R_1+2R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -11 & -7 & 2 & 0 \\ 0 & 1 & -4 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1+11R_3 \\ R_2+4R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 26 & 2 & 11 \\ 8 & 1 & 4 \\ 3 & 0 & 1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$\begin{aligned} \bar{x} &= A^{-1}\vec{b} \\ &= \begin{bmatrix} 26 & 2 & 11 \\ 8 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 16 \\ 45 \\ -45 \end{bmatrix} \\ &= \begin{bmatrix} 11 \\ -7 \\ 3 \end{bmatrix} \end{aligned}$$

6. [6 marks] Find k so that $\mathbf{w} = \begin{bmatrix} 21 \\ -22 \\ k \end{bmatrix}$ is in span($\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix}$).

Let $c_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -22 \\ K \end{bmatrix}$

$$\begin{array}{cc|c} c_1 & c_2 & \\ 1 & 3 & 21 \\ 3 & -8 & -22 \\ 4 & 3 & k \end{array}$$

$$\begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \end{matrix} \left[\begin{array}{cc|c} 1 & 3 & 21 \\ 0 & -17 & -85 \\ 0 & -9 & K-84 \end{array}\right]$$

$$\frac{R_2}{-17} \left[\begin{array}{cc|c} 1 & 3 & 21 \\ 0 & 1 & 5 \\ 0 & -9 & k-84 \end{array}\right]$$

$$R_3 + 9R_2 \left[\begin{array}{cc|c} 1 & 3 & 21 \\ 0 & 1 & 5 \\ 0 & 0 & K-39 \end{array}\right] \text{REF}$$

Solvable system

$\Rightarrow K - 39 = 0$

$\Rightarrow K = 39$