1. [4 marks] We want to balance:

$Al_2O_3 + Fe \rightarrow Fe_3O_4 + Al$

Set up a system of equations. DO NOT SOLVE THE SYSTEM.

$$w\, Al_2O_3 + x\, Fe \rightarrow y\, Fe_3O_4 + z\, Al$$

Al: $2w = z$

O: $3w = 4y$

Fe: $x = 3y$

or

$$\begin{aligned} 2w - z &= 0 \\ 3w - 4y &= 0 \\ x - 3y &= 0 \end{aligned}$$

or

$$\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -4 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{array}\right] \end{array}$$

2. [4 marks] B is an invertible matrix. Solve for X:
$(7I + BX)^T = A$

$$((7I + BX)^T)^T = A^T$$

$$7I + BX = A^T$$

$$BX = A^T - 7I$$

$$X = B^{-1}(A^T - 7I)$$

3. [1 mark] The set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly independent. Is the set of vectors $\{\mathbf{a}, \mathbf{b}\}$ linearly independent? Explain briefly.

Yes.

No linear dependency among $\vec{a}, \vec{b}, \vec{c}$

$\Rightarrow$ No linear dependency among $\vec{a}$ and $\vec{b}$

4. [4 marks] Given $A = \begin{bmatrix} -3 & 2 \\ 7 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Find $(A - 3B)^T C^2$.

$$= \left(\begin{bmatrix} -3 & 2 \\ 7 & 4 \end{bmatrix} - 3 \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix} \right)^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2$$

$$= \left(\begin{bmatrix} -3 & 2 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 12 \\ -12 & -3 \end{bmatrix} \right)^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 14 \\ -5 & 1 \end{bmatrix}^T \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -5 \\ 14 & 1 \end{bmatrix} \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} -117 & -170 \\ 113 & 162 \end{bmatrix}$$

5. [6 marks] Solve the system by finding A^{-1}.

$$\begin{aligned} x - 2y - \ \ 3z &= 15 \\ 4x - 7y - 16z &= 42 \\ -3x + 6y + 10z &= -42 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ 4 & -7 & -16 & 0 & 1 & 0 \\ -3 & 6 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-4R_1 \\ R_3+3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -3 & 1 & 0 & 0 \\ 0 & 1 & -4 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1+2R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -11 & -7 & 2 & 0 \\ 0 & 1 & -4 & -4 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1+11R_3 \\ R_2+4R_3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 26 & 2 & 11 \\ 8 & 1 & 4 \\ 3 & 0 & 1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$\begin{aligned} \vec{x} &= A^{-1}\vec{b} \\ &= \begin{bmatrix} 26 & 2 & 11 \\ 8 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 42 \\ -42 \end{bmatrix} \\ &= \begin{bmatrix} 12 \\ -6 \\ 3 \end{bmatrix} \end{aligned}$$

6. [6 marks] Find k so that $\mathbf{w} = \begin{bmatrix} 17 \\ -17 \\ k \end{bmatrix}$ is in span($\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix}$).

Let $c_1 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -8 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ -17 \\ k \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 3 & 17 \\ 3 & -8 & -17 \\ 4 & 3 & k \end{array}\right]$$

$$\begin{array}{l} \\ R_2-3R_1 \\ R_4-4R_1 \end{array} \left[\begin{array}{cc|c} 1 & 3 & 17 \\ 0 & -17 & -68 \\ 0 & -9 & k-68 \end{array}\right]$$

$$\frac{R_2}{-17} \left[\begin{array}{cc|c} 1 & 3 & 17 \\ 0 & 1 & 4 \\ 0 & -9 & k-68 \end{array}\right]$$

$$R_3+9R_2 \left[\begin{array}{cc|c} 1 & 3 & 17 \\ 0 & 1 & 4 \\ 0 & 0 & k-32 \end{array}\right] \text{REF}$$

Solvable system

$\Rightarrow k-32=0$

$\Rightarrow k=32$