

(7) Find the best-fit line
 $y = a_0 + a_1 x$

$$1(a_0) + x(a_1) = y$$

$$\begin{array}{c|cc} & a_0 & a_1 & y \\ \hline 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 3 & 1 & 3 & 5 \end{array}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \end{aligned}$$

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix}$$

$$\begin{aligned} \vec{a}^* &= (A^T A)^{-1} A^T \vec{b} \\ &= \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 14 & -6 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 17 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} -18 \\ 15 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 2.5 \end{bmatrix} \begin{array}{l} \leftarrow a_0 \\ \leftarrow a_1 \end{array} \end{aligned}$$

$$y = -3 + 2.5x$$

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Error vector $\bar{b} - A\bar{x}^* = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2.5 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -0.5 \\ 2 \\ 4.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}$$

Error $\| \bar{b} - A\bar{x}^* \| = \sqrt{0.5^2 + (-1)^2 + 0.5^2}$

$$\approx 1.225$$

(17)

Find the best-fit parabola

$$y = a_0 + a_1 x + a_2 x^2$$

$$a_0(1) + a_1(x) + a_2(x^2) = y$$

$$\begin{array}{ccc|c} 1 & x & x^2 & \\ \hline 1 & -2 & 4 & 4 \\ 1 & -1 & 1 & 7 \\ 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & -1 \end{array}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$

$$\text{Find } (A^T A)^{-1} = \begin{bmatrix} 5 & 0 & 10 & | & 1 & 0 & 0 \\ 0 & 10 & 0 & | & 0 & 1 & 0 \\ 10 & 0 & 34 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{R_1}{5} \begin{bmatrix} 1 & 0 & 2 & | & \frac{1}{5} & 0 & 0 \\ 0 & 10 & 0 & | & 0 & 1 & 0 \\ 10 & 0 & 34 & | & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 - 10R_1 \begin{bmatrix} 1 & 0 & 2 & | & \frac{1}{5} & 0 & 0 \\ 0 & 10 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 14 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\frac{R_2}{10} \begin{bmatrix} 1 & 0 & 2 & | & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 14 & | & -2 & 0 & 1 \end{bmatrix} \rightarrow$$

$$\begin{array}{l}
 R_3 \\
 T_4 \\
 R_1 - 2R_3
 \end{array}
 \left[\begin{array}{ccc|ccc}
 1 & 0 & 2 & \frac{1}{5} & 0 & 0 \\
 0 & 1 & 0 & 0 & \frac{1}{10} & 0 \\
 0 & 0 & 1 & -\frac{1}{7} & 0 & \frac{1}{4}
 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc}
 1 & 0 & 0 & \frac{17}{35} & 0 & -\frac{1}{7} \\
 0 & 1 & 0 & 0 & \frac{1}{10} & 0 \\
 0 & 0 & 1 & -\frac{1}{7} & 0 & \frac{1}{4}
 \end{array} \right]$$

$$\underbrace{\hspace{10em}}_{(A^T A)^{-1}}$$

$$\begin{aligned}
 \vec{x}^* &= (A^T A)^{-1} A^T \vec{b} \\
 &= \begin{bmatrix} \frac{17}{35} & 0 & -\frac{1}{7} \\ 0 & \frac{1}{10} & 0 \\ -\frac{1}{7} & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{17}{35} & 0 & -\frac{1}{7} \\ 0 & \frac{1}{10} & 0 \\ -\frac{1}{7} & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 13 \\ -17 \\ 19 \end{bmatrix} \\
 &= \begin{bmatrix} 18/5 \\ -17/10 \\ -1/2 \end{bmatrix} \begin{array}{l} \leftarrow a_0 \\ \leftarrow a_1 \\ \leftarrow a_2 \end{array}
 \end{aligned}$$

$$y = a_0 + a_1 x + a_2 x^2$$

$$y = \frac{18}{5} - \frac{17}{10} x - \frac{1}{2} x^2$$

(21)

$$\text{Find } \vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 8 \\ 8 & 38 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{468} \begin{bmatrix} 38 & -8 \\ -8 & 14 \end{bmatrix}$$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$
$$= \frac{1}{468} \begin{bmatrix} 38 & -8 \\ -8 & 14 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & -3 & 5 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -2 \\ 4 \end{bmatrix}$$
$$= \frac{1}{468} \begin{bmatrix} 38 & -8 \\ -8 & 14 \end{bmatrix} \begin{bmatrix} 12 \\ -21 \end{bmatrix}$$
$$= \frac{1}{468} \begin{bmatrix} 624 \\ -390 \end{bmatrix}$$
$$= \frac{1}{6} \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 4/3 \\ -5/6 \end{bmatrix}$$