Section 5.4

① $|A-\lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix}$

$= (4-\lambda)^2 - 1$

$= \lambda^2 - 8\lambda + 15$

$= (\lambda-5)(\lambda-3)$

Set $|A-\lambda I| = 0$: $(\lambda-5)(\lambda-3) = 0$

$\lambda = 5, 3$

E_3 : $[A - 3I \mid \vec{0}]$

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$R_2 - R_1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\uparrow$

$x_2 = t$

$x_1 + x_2 = 0 \Rightarrow x_1 = -t$

$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ or $\begin{bmatrix} 1 \\ -1 \end{bmatrix} t$

Orthonormal basis for $E_3 = \{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$

E_5 : $[A - 5I \mid \vec{0}]$

$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$\frac{R_1}{-1}$ $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow$

$$R_2 - R_1 \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_2 = t$$

$$x_1 - x_2 = 0 \Rightarrow x_1 = t$$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

Orthonormal basis for $E_5 = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

3 $\quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & \sqrt{2} \\ \sqrt{2} & -\lambda \end{vmatrix}$

$$= (1-\lambda)(-\lambda) - 2$$

$$= \lambda^2 - \lambda - 2$$

$$= (\lambda - 2)(\lambda + 1)$$

Set $|A - \lambda I| = 0$: $(\lambda - 2)(\lambda + 1) = 0$

$$\lambda = -1, 2$$

E_{-1} : $[A + I \mid \bar{0}]$

$$\left[\begin{array}{cc|c} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 1 & 0 \end{array}\right] \rightarrow$$

③ Cont'd $\frac{R_1}{2}$ $\left[\begin{array}{cc|c} 1 & \frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & 1 & 0 \end{array}\right]$

$R_2 - \sqrt{2}R_1$ $\left[\begin{array}{cc|c} 1 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\uparrow$

$x_2 = t$

$x_1 + \frac{\sqrt{2}}{2} x_2 = 0 \Rightarrow x_1 = \frac{-\sqrt{2}}{2} t$

$\bar{x} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 1 \end{bmatrix} t$ or $\begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix} t$

$\left\| \begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix} \right\| = \sqrt{2+4} = \sqrt{6}$

Normalize : $\frac{1}{\sqrt{6}} \begin{bmatrix} -\sqrt{2} \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$

or $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}$

Orthonormal basis for $E_{-1} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix} \right\}$

E_2 : $[A - 2I \mid \vec{0}]$

$\left[\begin{array}{cc|c} -1 & \sqrt{2} & 0 \\ \sqrt{2} & -2 & 0 \end{array}\right]$

$\frac{R_1}{-1}$ $\left[\begin{array}{cc|c} 1 & -\sqrt{2} & 0 \\ \sqrt{2} & -2 & 0 \end{array}\right] \rightarrow$

(3) Cont'd

$$R_2 - \sqrt{2}R_1 \quad \left[\begin{array}{cc|c} 1 & -\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 - \sqrt{2}x_2 = 0 \quad \Rightarrow \quad x_1 = \sqrt{2}t$

$$\bar{x} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} t$$

$$\left\| \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \right\| = \sqrt{2+1} = \sqrt{3}$$

Normalize : $\frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$ or $\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

Orthonormal basis for $E_2 = \left\{ \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{3} \end{bmatrix} \right\}$

$$Q = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

(5) $|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 3 & 1-\lambda \end{vmatrix}$

$= (5-\lambda) \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix}$

$= (5-\lambda)\left[(1-\lambda)^2 - 9\right]$

$= (5-\lambda)\left[\lambda^2 - 2\lambda - 8\right]$

$= (5-\lambda)(\lambda-4)(\lambda+2)$

Set $|A - \lambda I| = 0$: $\lambda = 5, 4, -2$

(5) Cont'd

E_5: $[A - 5I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 3 & 0 \\ 0 & 3 & -4 & 0 \end{array}\right]$$

$\frac{R_3}{3}$ then reorder rows
$$\left[\begin{array}{ccc|c} 0 & 1 & -\frac{4}{3} & 0 \\ 0 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 + 4R_1$
$$\left[\begin{array}{ccc|c} 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & -\frac{7}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-\frac{3}{7}R_2$
$$\left[\begin{array}{ccc|c} 0 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 + \frac{4}{3}R_2$
$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = t \qquad x_2 = 0, \quad x_3 = 0$

$$\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$$

orthonormal basis for $E_5 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

E_4: $[A - 4I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$\frac{R_2}{-3}$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \rightarrow$$

(5) Cont'd

$$R_3 - 3R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_3 = t$$

$$x_1 = 0$$
$$x_2 - x_3 = 0 \Rightarrow x_2 = t$$

$$\bar{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t$$

$$\left\| \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{2}$$

Normalize: $\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Orthonormal basis for $E_4 = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

E_{-2}: $\quad [A + 2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 7 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$\frac{R_1}{7}$ and $\frac{R_2}{3}$ $\quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$

$R_2 - 3R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$\uparrow$$

$$x_3 = t$$

$$x_1 = 0$$
$$x_2 + x_3 = 0 \Rightarrow x_2 = -t$$

$$\bar{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t \qquad \left\| \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{2}$$

$\rightarrow$

(5) Cont'd

Normalize: $\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$

Orthonormal basis for $E_{-2} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

(13 a) A matrix M is orthogonally diagonalizable if and only if $M^T = M$.

A and B orthogonally diagonalizable

$\Rightarrow A^T = A$ and $B^T = B$

$$\Rightarrow (A+B)^T = A^T + B^T$$
$$= A + B$$

$\Rightarrow A+B$ is orthogonally diagonalizable

See Question 1.

(17) Orthonormal eigenvectors in column form:

$$\vec{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 = 3 \qquad \lambda_2 = 5$$

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T}$$

$$= 3 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} + 5 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

This is the "spectral decomposition."

(19) See Question 5.

Orthonormal eigenvectors in column form:

$$\vec{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 = 5 \qquad \lambda_2 = 4 \qquad \lambda_3 = -2$$

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \lambda_3 \vec{q}_3 \vec{q}_3^{\,T}$$

$$= 5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} - 2 \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

This is the "spectral decomposition."

(21) Orthonormal eigenvectors in column form:

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_1 = -1 \qquad \lambda_2 = 2$$

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T}$$

$$= -\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} + 2 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$= \frac{-1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{2}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$$

(23) Orthonormal eigenvectors in column form:

$$\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} \quad \vec{q}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} \quad \vec{q}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3$$

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \lambda_3 \vec{q}_3 \vec{q}_3^{\,T}$$

$$= 1 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix}1\\1\\0\end{bmatrix}\begin{bmatrix}1 & 1 & 0\end{bmatrix} + 2 \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \begin{bmatrix}1\\-1\\1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\end{bmatrix}$$

$$+ 3 \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{bmatrix}-1\\1\\2\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{bmatrix} + \frac{2}{3}\begin{bmatrix}1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 1\end{bmatrix} + \frac{3}{6}\begin{bmatrix}1 & -1 & -2\\-1 & 1 & 2\\-2 & 2 & 4\end{bmatrix}$$

$$= \frac{3}{6}\begin{bmatrix}1 & 1 & 0\\1 & 1 & 0\\0 & 0 & 0\end{bmatrix} + \frac{4}{6}\begin{bmatrix}1 & -1 & 1\\-1 & 1 & -1\\1 & -1 & 1\end{bmatrix} + \frac{3}{6}\begin{bmatrix}1 & -1 & -2\\-1 & 1 & 2\\-2 & 2 & 4\end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix}10 & -4 & -2\\-4 & 10 & 2\\-2 & 2 & 16\end{bmatrix}$$

$$\text{or} \quad \frac{1}{3}\begin{bmatrix}5 & -2 & -1\\-2 & 5 & 1\\-1 & 1 & 8\end{bmatrix}$$