① $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$2\bar{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Orthogonal Basis $= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

Orthonormal Basis $= \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

③ $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$

$$\bar{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} + \frac{6}{3} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\bar{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \rightarrow$$

$$\vec{v}_3 = \begin{bmatrix}3\\2\\4\end{bmatrix} + \frac{3}{3}\begin{bmatrix}1\\-1\\-1\end{bmatrix} - \frac{12}{6}\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}3\\2\\4\end{bmatrix} + \begin{bmatrix}1\\-1\\-1\end{bmatrix} - 2\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

Orthogonal basis $= \left\{ \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}$

Orthonormal basis $= \left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\-1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}2\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix} \right\}$

(5) $\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ Partial Basis $X = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix}3\\4\\2\end{bmatrix} - \mathrm{proj}_X \begin{bmatrix}3\\4\\2\end{bmatrix}$$

$$= \begin{bmatrix}3\\4\\2\end{bmatrix} - \frac{7}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$2\vec{v}_2 = \begin{bmatrix}6\\8\\4\end{bmatrix} - 7\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}-1\\1\\4\end{bmatrix}$$

Orthogonal basis $= \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\4\end{bmatrix} \right\}$

Orthonormal basis $= \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{18}}\begin{bmatrix}-1\\1\\4\end{bmatrix} \right\}$

Note: Any ~~nonzero~~ multiple of the basis vectors is also acceptable for the orthogonal basis.

(7) From Question 5, an orthogonal basis for W is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\}$ and $\bar{v} = \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix}$

$$\text{proj}_W \bar{v} = \frac{\bar{v} \cdot \vec{w}_1}{\|\vec{w}_1\|^2} \vec{w}_1 + \frac{\bar{v} \cdot \vec{w}_2}{\|\vec{w}_2\|^2} \vec{w}_2$$

$$= \frac{0}{2} \vec{w}_1 + \frac{4}{18} \vec{w}_2$$

$$= \frac{2}{9} \vec{w}_2$$

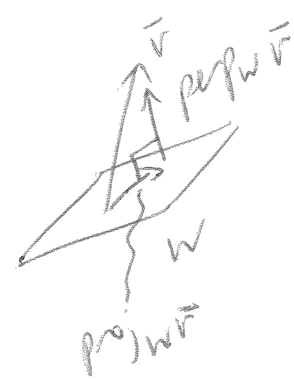
$$= \begin{bmatrix} -2/9 \\ 2/9 \\ 8/9 \end{bmatrix}$$

$$\text{perp}_W \bar{v} = \bar{v} - \text{proj}_W \bar{v}$$

$$= \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/9 \\ 2/9 \\ 8/9 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 36 \\ -36 \\ 27 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ -8 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 38 \\ -38 \\ 19 \end{bmatrix}$$



(9) Find an orthogonal basis for

$$\operatorname{span}\left(\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right)$$

$$\bar{v}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix} \qquad \text{Partial Basis } X = \left\{\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

$$\bar{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix} - \operatorname{proj}_X \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$2\bar{v}_2 = \begin{bmatrix}2\\0\\2\end{bmatrix} - \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\-1\\1\end{bmatrix} \qquad \text{Partial Basis } X = \left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$$

$$\bar{v}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \operatorname{proj}_X \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{6}\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$$6\bar{v}_3 = \begin{bmatrix}6\\6\\0\end{bmatrix} - \begin{bmatrix}0\\3\\3\end{bmatrix} - \begin{bmatrix}2\\-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}4\\4\\-4\end{bmatrix}$$

Orthogonal basis for $\operatorname{Col}(A) = \left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}4\\4\\-4\end{bmatrix}\right\}$

<u>OR</u>: Any nonzero multiples of these vectors.

(11) Start with any basis for $\mathbb{R}^3$ containing $\begin{bmatrix}3\\1\\5\end{bmatrix}$

Say $\left\{ \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$

We know it's a basis for $\mathbb{R}^3$ because $\begin{vmatrix}3 & 1 & 0\\1 & 0 & 1\\5 & 0 & 0\end{vmatrix} \neq 0.$

$\vec{v}_1 = \begin{bmatrix}3\\1\\5\end{bmatrix}$ Partial Basis $X = \left\{ \begin{bmatrix}3\\1\\5\end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{3}{35}\begin{bmatrix}3\\1\\5\end{bmatrix}$$

$$35\vec{v}_2 = 35\begin{bmatrix}1\\0\\0\end{bmatrix} - 3\begin{bmatrix}3\\1\\5\end{bmatrix}$$

$$= \begin{bmatrix}26\\-3\\-15\end{bmatrix}$$ Partial Basis $X = \left\{ \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}26\\-3\\-15\end{bmatrix} \right\}$

$$\vec{v}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \text{proj}_X \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{35}\begin{bmatrix}3\\1\\5\end{bmatrix} + \frac{3}{910}\begin{bmatrix}26\\-3\\-15\end{bmatrix}$$

$$910\vec{v}_3 = 910\begin{bmatrix}0\\1\\0\end{bmatrix} - 26\begin{bmatrix}3\\1\\5\end{bmatrix} + 3\begin{bmatrix}26\\-3\\-15\end{bmatrix}$$

$$= \begin{bmatrix}0\\875\\-175\end{bmatrix}$$

Orthogonal basis $= \left\{ \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}26\\-3\\-15\end{bmatrix}, \begin{bmatrix}0\\875\\-175\end{bmatrix} \right\}$ or any nonzero multiples of these vectors.

<u>Note</u>: Different starting bases will produce different answers.

(13) Let $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & a \\ 0 & 1/\sqrt{3} & b \\ -1/\sqrt{2} & 1/\sqrt{3} & c \end{bmatrix}$

Columns 1 and 3 are orthogonal $\Rightarrow$ $\frac{a}{\sqrt{2}} - \frac{c}{\sqrt{2}} = 0$

$$a - c = 0$$
$$c = a$$

Columns 2 and 3 are orthogonal $\Rightarrow$ $\frac{a + b + c}{\sqrt{3}} = 0$

$$a + b + c = 0$$

$c = a \rightarrow$ $2a + b = 0$

$$b = -2a$$

Last column has the ~~form~~ $\begin{bmatrix} a \\ -2a \\ a \end{bmatrix}$.

Last column is a multiple of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Last column $= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

(15) See Question 9

Orthogonal basis for $Col(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix} \right\}$

Orthonormal basis for $Col(A) = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$$Q = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$A = QR$$

$$Q^T A = \underbrace{Q^T Q}_{I} R$$

$$R = Q^T A$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

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$$A = QR$$

$$Q^T A = \underbrace{Q^T Q}_{I} R$$

$$R = Q^T A$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 8 & 2 \\ 1 & 7 & -1 \\ -2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9 & 27 & 1 \\ 0 & 18 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$