Section 5.2

(3) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$

Find a basis for W

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \end{array}\right] \end{array}$$

$y = s$

$z = t$

$x + y - z = 0 \Rightarrow x = -s + t$

$$\bar{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $W = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for $W^{\perp}$:

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$\overset{\frac{R_1}{-1}}{=} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$R_1 + R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$x_3 = t$

$x_1 + x_3 = 0 \Rightarrow x_1 = -t$

$x_2 + x_3 = 0 \Rightarrow x_2 = -t$

$\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} t$ or $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} t$ Basis for $W^{\perp} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

7 $\quad A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$$\begin{array}{l} \\ R_2 - 5R_1 \\ \\ R_4 + R_1 \end{array} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 7 & -14 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 7 & -14 \\ 0 & -2 & 4 \end{bmatrix}$$

$$\begin{array}{l} R_1 + R_2 \\ \\ R_3 - 7R_2 \\ R_4 + 2R_2 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{RREF}$$

Basis for row(A) = nonzero rows of REF/RREF

$= \{[1 \; 0 \; 1], [0 \; 1 \; -2]\}$

Basis for null(A): Solve $A\vec{x} = \vec{0}$

$$\left[ A \mid \vec{0} \right]$$

Via row operations above: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 = t$

$x_1 + x_3 = 0 \Rightarrow x_1 = -t$

$x_2 - 2x_3 = 0 \Rightarrow x_2 = 2t$

$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} t$

Basis for null(A) $= \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

Note: Coursepack answer is <u>incorrect</u>.

Confirm that $[1 \; 0 \; 1] \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0$ ✓ $\quad [0 \; 1 \; -2] \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 0$ ✓

(9) See Question 7 for RREF of A

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ RREF}$$

Basis for col(A) = Columns 1 and 2 of A

$$= \left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Basis for null $(A^T)$ : Solve $A^T \bar{x} = \bar{0}$

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & -1 & 0 \\ -1 & 2 & 1 & -1 & 0 \\ 3 & 1 & -2 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 5 & 0 & -1 & 0 \\ 0 & 7 & 1 & -2 & 0 \\ 0 & -14 & -2 & 4 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ \frac{R_2}{7} \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 5 & 0 & -1 & 0 \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & -14 & -2 & 4 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - 5R_2 \\ \\ R_3 + 14R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{5}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cc} \uparrow & \uparrow \\ x_3 = s & x_4 = t \end{array} \rightarrow$$

⑨ Cont'd

$x_1 - \frac{5}{7}x_3 + \frac{3}{7}x_4 = 0 \Rightarrow x_1 = \frac{5}{7}s - \frac{3}{7}t$

$x_2 + \frac{1}{7}x_3 - \frac{2}{7}x_4 = 0 \Rightarrow x_2 = \frac{-1}{7}s + \frac{2}{7}t$

$$\bar{x} = \begin{bmatrix} 5/7 \\ -1/7 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3/7 \\ 2/7 \\ 0 \\ 1 \end{bmatrix} t$$

$$\text{or} \quad \begin{bmatrix} 5 \\ -1 \\ 7 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 2 \\ 0 \\ 7 \end{bmatrix} t$$

Basis for $\text{null}(A^T) = \left\{ \begin{bmatrix} 5 \\ -1 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 7 \end{bmatrix} \right\}$

Call the basis for col (A) : $\bar{c}_1, \bar{c}_2$
" " null ($A^T$) : $\bar{n}_1, \bar{n}_2$

Check that $\bar{c}_1 \cdot \bar{n}_1 = 0$, $\bar{c}_1 \cdot \bar{n}_2 = 0$, $\bar{c}_2 \cdot \bar{n}_1 = 0$, $\bar{c}_2 \cdot \bar{n}_2 = 0$ ✓

<u>Note</u> : Coursepack gives a different but correct answer for basis for null($A^T$).

(13)

Solve

$$\left[\begin{array}{cccc|c} 2 & -1 & 6 & 3 & 0 \\ -1 & 2 & -3 & -2 & 0 \\ 2 & 5 & 6 & 1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ then $\frac{R_1}{-1}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 2 & 0 \\ 2 & -1 & 6 & 3 & 0 \\ 2 & 5 & 6 & 1 & 0 \end{array}\right]$$

$R_2 - 2R_1$
$R_3 - 3R_1$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 2 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 9 & 0 & -3 & 0 \end{array}\right]$$

$\frac{R_2}{3}$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & 2 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 9 & 0 & -3 & 0 \end{array}\right]$$

$R_1 + 2R_2$

$R_3 - 9R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$ $\quad$ $x_4 = t$

$$x_1 + 3x_3 + \frac{4}{3}x_4 = 0 \Rightarrow x_1 = -3s - \frac{4}{3}t$$

$$x_2 - \frac{1}{3}x_4 = 0 \Rightarrow x_2 = \frac{1}{3}t$$

$$\vec{x} = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix} t$$

or $\vec{x} = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} t$

Basis for $W^{\perp} = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}$

(17) Check that $\bar{u}_1$ and $\bar{u}_2$ are orthogonal:

$$\bar{u}_1 \cdot \bar{u}_2 = 0 \checkmark$$

$$\text{proj}_W \bar{v} = \text{proj}_{\vec{u}_1} \bar{v} + \text{proj}_{\bar{u}_2} \vec{v}$$

$\boxed{\text{because basis for } W \text{ is orthogonal}}$

$$= \frac{\bar{u}_1 \cdot \bar{v}}{\|\bar{u}_1\|^2} \bar{u}_1 + \frac{\bar{u}_2 \cdot \bar{v}}{\|\vec{u}_2\|^2} \bar{u}_2$$

$$= \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{13}{18} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{2}{18} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{13}{18} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{18} \begin{bmatrix} -9 \\ 9 \\ 54 \end{bmatrix}$$

or $\begin{bmatrix} -1/2 \\ 1/2 \\ 3 \end{bmatrix}$

(21) Check that basis for W is orthogonal:

$$\begin{bmatrix}1\\2\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\1\end{bmatrix} = 0 \checkmark$$

$$\text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v}$$

$\boxed{\text{because basis for W is orthogonal}}$

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\|\vec{w}_1\|^2} \vec{w}_1 + \frac{\vec{w}_2 \cdot \vec{v}}{\|\vec{w}_2\|^2} \vec{w}_2$$

$$= \frac{3}{6} \begin{bmatrix}1\\2\\1\end{bmatrix} + \frac{9}{3} \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix}1\\2\\1\end{bmatrix} + \frac{6}{2} \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix}7\\-4\\7\end{bmatrix}$$

$$= \begin{bmatrix}7/2\\-2\\7/2\end{bmatrix}$$

$$\text{perp}_W \vec{v} = \vec{v} - \text{proj}_W \vec{v}$$

$$= \begin{bmatrix}4\\-2\\3\end{bmatrix} - \begin{bmatrix}7/2\\-2\\7/2\end{bmatrix}$$

$$= \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$$